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Abelian and nonabelian vector field effective actions from string field theory

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Abstract: The leading terms in the tree-level effective action for the massless fields of the bosonic open string are calculated by integrating out all massive fields in Witten's cubic string field theory. In both the abelian and nonabelian theories, field redefinitions make it possible to express the effective action in terms of the conventional field strength. The resulting actions reproduce the leading terms in the abelian and nonabelian Born-Infeld theories. In principle this method can be used to systematically determine all terms in the abelian and nonabelian actions, including all higher-order derivative terms.

Keywords: Bosonic Strings, D-branes.

Contents

1. Introduction

Despite major advances in our understanding of nonperturbative features of string theory and M-theory over the last eight years, we still lack a fundamental nonperturbative and background-independent definition of string theory. String field theory seems to incorporate some features of background independence which are missing in other approaches to string theory. Recent work, following the conjectures of Sen [1], has shown that Witten's open bosonic string field theory successfully describes multiple distinct open string vacua with dramatically different geometrical properties, in terms of the degrees of freedom of a single theory (see [2]–[5] for reviews of this work). An important feature of string field theory, which allows it to transcend the usual limitations of local quantum field theories, is its essential nonlocality. String field theory is a theory which can be defined with reference to a particular background in terms of an infinite number of space-time fields, with highly nonlocal interactions. The nonlocality of string field theory is similar in spirit to that of noncommutative field theories which have been the subject of much recent work [6], but in string field theory the nonlocality is much more extreme. In order to understand how string theory encodes a quantum theory of gravity at short distance scales, where geometry becomes poorly defined, it is clearly essential to achieve a better understanding of the nonlocal features of string theory.

While string field theory involves an infinite number of space-time fields, most of these fields have masses on the order of the Planck scale. By integrating out the massive fields, we arrive at an effective action for a finite number of massless fields. In the case of a closed string field theory, performing such an integration would give an effective action for the usual multiplet of gravity/supergravity fields. This action will, however, have a complicated nonlocal structure which will appear through an infinite family of higher-derivative terms in the effective action. In the case of the open string, integrating out the massive fields leads to an action for the massless gauge field. Again, this action is highly nonlocal and contains an infinite number of higher-derivative terms. This nonlocal action for the massless gauge field in the bosonic open string theory is the subject of this paper. By explicitly integrating out all massive fields in Witten's open string field theory (including the tachyon), we arrive at an effective action for the massless open string vector field. We compute this effective action term-by-term using the level-truncation approximation in string field theory, which gives us a very accurate approximation to each term in the action.

It is natural to expect that the effective action we compute for the massless vector field will take the form of the Born-Infeld action, including higher-derivative terms. Indeed, we show that this is the case, although some care must be taken in making this connection. Early work deriving the Born-Infeld action from string theory [7, 8] used world-sheet methods [9, 10]. More recently, in the context of the supersymmetric nonabelian gauge field action, other approaches, such as κ-symmetry and the existence of supersymmetric solutions, have been used to constrain the form of the action (see [11] for a recent discussion and further references). In this work we take a different approach. We start with string field theory, which is a manifestly off-shell formalism. Our resulting effective action is therefore also an off-shell action. This action has a gauge invariance which agrees with the usual

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Yang-Mills gauge invariance to leading order, but which has higher-order corrections arising from the string field star product. A field redefinition analogous to the Seiberg-Witten map [12, 13] is necessary to get a field which transforms in the usual fashion [14, 15]. We identify the leading terms in this transformation and show that after performing the field redefinition our action indeed takes the Born-Infeld form in the abelian theory. In the non-abelian theory, there is an additional subtlety, which was previously encountered in related contexts in [14, 15]. Extra terms appear in the form of the gauge transformation which cannot be removed by a field redefinition. These additional terms, however, are trivial and can be dropped, after which the standard form of gauge invariance can be restored by a field redefinition. This leads to an effective action in the nonabelian theory which takes the form of the nonabelian Born-Infeld action plus derivative correction terms.

It may seem surprising that we integrate out the tachyon as well as the fields in the theory with positive mass squared. This is, however, what is implicitly done in previous work such as [7, 8] where the Born-Infeld action is derived from bosonic string theory. The abelian Born-Infeld action can similarly be derived from recent proposals for the coupled tachyon-vector field action [16]–[19] by solving the equation of motion for the tachyon at the top of the hill. In the supersymmetric theory, of course, there is no tachyon on a BPS brane, so the supersymmetric Born-Infeld action should be derivable from a supersymmetric open string field theory by only integrating out massive fields. Physically, integrating out the tachyon corresponds to considering fluctuations of the D-brane in stable directions, while the tachyon stays balanced at the top of its potential hill. While open string loops may give rise to problems in the effective theory [20], at the classical level the resulting action is well-defined and provides us with an interesting model in which to understand the nonlocality of the Born-Infeld action. The classical effective action we derive here must reproduce all on-shell tree-level scattering amplitudes of massless vector fields in bosonic open string theory. To find a sensible action which includes quantum corrections, it is probably necessary to consider the analogue of the calculation in this paper in the supersymmetric theory, where there is no closed string tachyon.

The structure of this paper is as follows: In section 2 we review the formalism of string field theory, set notation and make some brief comments regarding the Born-Infeld action. In section 3 we introduce the tools needed to calculate terms in the effective action of the massless fields. Section 4 contains a calculation of the effective action for all terms in the Yang-Mills action. Section 5 extends the analysis to include the next terms in the Born-Infeld action in the abelian case and section 6 does the same for the nonabelian analogue of the Born-Infeld action. Section 7 contains concluding remarks. Some useful properties of the Neumann matrices appearing in the 3-string vertex of Witten's string field theory are included in the appendix.

2. Review of formalism

Subsection 2.1 summarizes our notation and the basics of string field theory. In subsection 2.2 we review the method of [21] for computing terms in the effective action. The last subsection, 2.3, contains a brief discussion of the Born-Infeld action.

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2.1 Basics of string field theory

In this subsection we review the basics of Witten's open string field theory [22]. For further background information see the reviews [23, 24, 25, 4]. The degrees of freedom of string field theory (SFT) are functionals $\Phi[x(\sigma); c(\sigma), b(\sigma)]$ of the string configuration $x^\mu(\sigma)$ and the ghost and antighost fields $c(\sigma)$ and $b(\sigma)$ on the string at a fixed time. String functionals can be expressed in terms of string Fock space states, just as functions in $\mathcal{L}^2(\mathbb{R})$ can be expressed as linear combinations of harmonic oscillator eigenstates. The Fock module of a single string of momentum p is obtained by the action of the matter, ghost and antighost oscillators on the (ghost number one) highest weight vector $|p\rangle$. The action of the raising and lowering oscillators on $|p\rangle$ is defined by the creation/annihilation conditions and commutation relations

$$\begin{aligned} a^\mu_{n\geq 1}|p\rangle &= 0\,, & [a^\mu_m, a^\nu_{-n}] &= \eta^{\mu\nu}\delta_{m,n}\,, \\ p^\mu|k\rangle &= k^\mu|k\rangle\,, & & \\ b_{n\geq 0}|p\rangle &= 0\,, & \{b_m, c_{-n}\} &= \delta_{m,n}\,, \\ c_{n\geq 1}|p\rangle &= 0\,. & & \end{aligned} \tag{2.1}$$

Hermitian conjugation is defined by $a^{\mu\dagger}_n = a^\mu_{-n}$, $b^\dagger_n = b_{-n}$, $c^\dagger_n = c_{-n}$. The single-string Fock space is then spanned by the set of all vectors $|\chi\rangle = \cdots a_{n_2}a_{n_1}\cdots b_{k_2}b_{k_1}\cdots c_{l_2}c_{l_1}|p\rangle$ with $n_i, k_i < 0$ and $l_i \leq 0$. String fields of ghost number 1 can be expressed as linear combinations of such states $|\chi\rangle$ with equal number of b's and c's, integrated over momentum.

$$|\Phi\rangle = \int d^{26}p\ \left(\phi(p) + A_\mu(p)\ a^\mu_{-1} - i\alpha(p)b_{-1}c_0 + B_{\mu\nu}(p)a^\mu_{-1}a^\nu_{-1} + \cdots\right)|p\rangle\,. \tag{2.2}$$

The Fock space vacuum $|0\rangle$ that we use is related to the SL$(2,\mathbb{R})$ invariant vacuum $|1\rangle$ by $|0\rangle = c_1|1\rangle$. Note that $|0\rangle$ is a Grassmann odd object, so that we should change the sign of our expression whenever we interchange $|0\rangle$ with a Grassmann odd variable. The bilinear inner product between the states in the Fock space is defined by the commutation relations and

$$\langle k|\, c_0\, |p\rangle = (2\pi)^{26}\delta(k+p)\,. \tag{2.3}$$

The SFT action can be written as

$$S = -\frac{1}{2}\langle V_2|\,\Phi, Q_B\Phi\rangle - \frac{g}{3}\langle V_3|\,\Phi,\Phi,\Phi\rangle \tag{2.4}$$

where $|V_n\rangle \in \mathcal{H}^n$. This action is invariant under the gauge transformation

$$\delta|\Phi\rangle = Q_B|\Lambda\rangle + g\big(\langle\Phi,\Lambda|\,V_3\rangle - \langle\Lambda,\Phi|\,V_3\rangle\big) \tag{2.5}$$

with Λ a string field gauge parameter at ghost number 0. Explicit oscillator representations of $\langle V_2|$ and $\langle V_3|$ are given by [26, 27, 28, 29]

$$\langle V_2| = \int d^{26}p\ \langle p|^{(1)}\otimes\langle -p|^{(2)}\left(c_0^{(1)} + c_0^{(2)}\right)\exp\left(a^{(1)}\cdot C\cdot a^{(2)} - b^{(1)}\cdot C\cdot c^{(2)} - b^{(1)}\cdot C\cdot c^{(2)}\right) \tag{2.6}$$

and

$$\langle V_3| = \mathcal{N} \int \prod_{i=1}^{3} \left(d^{26} p_i \langle p|^{(i)} c_0^{(i)} \right) \delta\left(\sum p_j \right) \times$$
$$\times \exp\left(\frac{1}{2} a^{(r)} \cdot V^{rs} \cdot a^{(s)} - p^{(r)} V_{0\cdot}^{rs} \cdot a^{(s)} + \frac{1}{2} p^{(r)} V_{00}^{rr} p^{(r)} - b^{(r)} \cdot X^{rs} \cdot c^{(s)} \right) \quad (2.7)$$

where all inner products denoted by $\cdot$ indicate summation from 1 to ∞ except in $b \cdot X$, where the summation includes the index 0. The contracted Lorentz indices in a_n^{μ} and p_{μ} are omitted. $C_{mn} = (-1)^n \delta_{mn}$ is the BPZ conjugation matrix. The matrix elements V_{mn}^{rs} and X_{mn}^{rs} are called Neumann coefficients. Explicit expressions for the Neumann coefficients and some relevant properties of these coefficients are summarized in the appendix. The normalization constant $\mathcal{N}$ is defined by

$$\mathcal{N} = \exp\left(-\frac{1}{2} \sum_r V_{00}^{rr} \right) = \frac{3^{9/2}}{2^6} , \quad (2.8)$$

so that the on-shell three-tachyon amplitude is given by $2g$. We use units where $\alpha' = 1$.

2.2 Calculation of effective action

String field theory can be thought of as a (nonlocal) field theory of the infinite number of fields that appear as coefficients in the oscillator expansion (2.2). In this paper, we are interested in integrating out all massive fields at tree level. This can be done using standard perturbative field theory methods. Recently an efficient method of performing sums over intermediate particles in Feynman graphs was proposed in [21]. We briefly review this approach here; an alternative approach to such computations has been studied recently in [30].

In this paper, while we include the massless auxiliary field α appearing in the expansion (2.2) as an external state in Feynman diagrams, all the massive fields we integrate out are contained in the Feynman-Siegel gauge string field satisfying

$$b_0 |\Phi\rangle = 0 , \quad (2.9)$$

This means that intermediate states in the tree diagrams we consider do not have a c_0 in their oscillator expansion. For such states, the propagator can be written in terms of a Schwinger parameter τ as

$$\frac{b_0}{L_0} = b_0 \int_0^{\infty} d\tau \, e^{-\tau L_0} , \quad (2.10)$$

In string field theory, the Schwinger parameters can be interpreted as moduli for the Riemann surface associated with a given diagram [31, 32, 33, 25, 34].

In field theory one computes amplitudes by contracting vertices with external states and propagators. Using the quadratic and cubic vertices (2.6), (2.7) and the propagator (2.10) we can do same in string field theory. To write down the contribution to the effective action arising from a particular Feynman graph we include a vertex $\langle V_3| \in \mathcal{H}^{*3}$ for each vertex of the graph and a vertex $|V_2\rangle$ for each internal edge. The propagator (2.10) can be incorporated into the quadratic vertex through[1]

$$|P\rangle = -\int_0^\infty d\tau\, e^{\tau(1-p^2)}\left|\tilde{V}_2\right\rangle . \tag{2.13}$$

where in the modified vertex $|\tilde{V}_2(\tau)\rangle$ the ghost zero modes c_0 are canceled by the b_0 in (2.10) and the matrix C_{mn} is replaced by

$$\tilde{C}_{mn}(\tau) = e^{-m\tau}(-1)^m \delta_{mn} . \tag{2.14}$$

With these conventions, any term in the effective action can be computed by contracting the three-vertices from the corresponding Feynman diagram on the left with factors of $|P\rangle$ and low-energy fields on the right (or vice-versa, with $|V_3\rangle$'s on the right and $\langle P|$'s on the left). Because the resulting expression integrates out *all* Feynman-Siegel gauge fields along interior edges, we must remove the contribution from the intermediate massless vector field by hand when we are computing the effective action for the massless fields. Note that in [21], a slightly different method was used from that just described; there the propagator was incorporated into the three-vertex rather than the two-vertex. Both methods are equivalent; we use the method just described for convenience.

States of the form

$$\exp\left(\lambda\cdot a^\dagger + \frac{1}{2}a^\dagger\cdot S\cdot a^\dagger\right)|p\rangle \tag{2.15}$$

are called squeezed states. The vertex $|V_3\rangle$ and the propagator $|P\rangle$ are (linear combinations of) squeezed states and thus are readily amenable to computations. The inner product of two squeezed states is given by [35]

$$\begin{aligned}
&\langle 0|\exp\left(\lambda\cdot a + \frac{1}{2}a\cdot S\cdot a\right)\exp\left(\mu\cdot a^\dagger + \frac{1}{2}a^\dagger\cdot V\cdot a^\dagger\right)|0\rangle = \\
&\quad = \mathrm{Det}(1-S\cdot V)^{-1/2}\exp\Big[\lambda\cdot(1-V\cdot S)^{-1}\cdot\mu + \frac{1}{2}\lambda\cdot(1-V\cdot S)^{-1}\cdot V\cdot\lambda + \\
&\qquad + \frac{1}{2}\mu\cdot S\cdot(1-V\cdot S)^{-1}\cdot\mu\Big]
\end{aligned} \tag{2.16}$$

and (neglecting ghost zero-modes)

$$\begin{aligned}
&\langle 0|\exp(b\cdot\lambda_b - \lambda_c\cdot c - b\cdot S\cdot c)\exp\left(b^\dagger\cdot\mu_b + \mu_c\cdot c^\dagger + b^\dagger\cdot V\cdot c^\dagger\right)|0\rangle = \\
&\quad = \mathrm{Det}(1-S\cdot V)\exp\Big[-\lambda_c\cdot(1-V\cdot S)^{-1}\cdot\mu_b - \mu_c\cdot(1-S\cdot V)^{-1}\cdot\lambda_b + \\
&\qquad + \lambda_c\cdot(1-V\cdot S)^{-1}\cdot V\cdot\lambda_b + \mu_c\cdot S\cdot(1-V\cdot S)^{-1}\cdot\mu_b\Big] .
\end{aligned} \tag{2.17}$$

[1]Consider the tachyon propagator as an example. We contract $c_0|p_1\rangle$ and $c_0|p_2\rangle$ with $\langle P|$ to get

$$\langle P|\, c_0|p_1\rangle c_0|p_2\rangle = -\int_0^\infty d\tau e^{\tau(1-p_1^2)}\delta(p_1+p_2) = -\frac{\delta(p_1+p_2)}{p_1^2-1} . \tag{2.11}$$

This formula assumes that both momenta are incoming. Setting $p_1 = -p_2 = p$ and using the metric with $(-,+,+,\ldots,+)$ signature we have

$$-\frac{1}{p^2+m^2} = \frac{1}{p_0^2-\vec{p}^2-m^2} \tag{2.12}$$

thus (2.11) is indeed the correct propagator for the scalar particle of mass $m^2=-1$.

Using these expressions, the combination of three-vertices and propagators associated with any Feynman diagram can be simply rewritten as an integral over modular (Schwinger) parameters of a closed form expression in terms of the infinite matrices $V_{nm}, X_{nm}, \tilde{C}_{nm}(\tau)$. The schematic form of these integrals is

$$
\begin{aligned}
(\langle V_3|)^v(|P\rangle)^i \sim & \left(\prod_{j=1}^{i} \int d\tau^j\right) \frac{\mathrm{Det}(1-\hat{C}\hat{X})}{\mathrm{Det}(1-\hat{C}\hat{V})^{13}} \times \\
& \times (\langle 0|)^{3v-2i} \exp\left(\frac{1}{2} a^\dagger \cdot S \cdot a^\dagger + \mu \cdot a^\dagger + b^\dagger \cdot U \cdot c^\dagger + \mu_c \cdot c^\dagger + b^\dagger \cdot \mu_b\right)
\end{aligned} \tag{2.18}
$$

where $\hat{C}, \hat{X}, \hat{V}$ are matrices with blocks of the form $\tilde{C}, X, V$ arranged according to the combinatorial structure of the diagram. The matrix $\hat{C}$ and the squeezed state coefficients S, U, μ, μ_b, μ_c depend implicitly on the modular parameters τ^i.

2.3 The effective vector field action and Born-Infeld

In this subsection we describe how the effective action for the vector field is determined from SFT and we discuss the Born-Infeld action [36] which describes the leading terms in this effective action. For a more detailed review of the Born-Infeld action, see [37]

As discussed in subsection 2.1, the string field theory action is a space-time action for an infinite set of fields, including the massless fields $A_\mu(x)$ and $\alpha(x)$. This action has a very large gauge symmetry, given by (2.5). We wish to compute an effective action for $A_\mu(x)$ which has a single gauge invariance, corresponding at leading order to the usual Yang-Mills gauge invariance. We compute this effective action in several steps. First, we use Feynman-Siegel gauge (2.9) for all *massive* fields in the theory. This leaves a single gauge invariance, under which A_μ and α have linear components in their gauge transformation rules. This partial gauge fixing is described more precisely in section 5.2. Following this partial gauge fixing, all massive fields in the theory, including the tachyon, can be integrated out using the method described in the previous subsection, giving an effective action

$$\check{S}[A_\mu(x), \alpha(x)] \tag{2.19}$$

depending on A_μ and α. We can then further integrate out the field α, which has no kinetic term, to derive the desired effective action

$$S[A_\mu(x)]\,. \tag{2.20}$$

The action (2.20) still has a gauge invariance, which at leading order agrees with the Yang-Mills gauge invariance

$$\delta A_\mu(x) = \partial_\mu \lambda(x) - i g_{\mathrm{YM}}[A_\mu(x), \lambda(x)] + \cdots\,. \tag{2.21}$$

The problem of computing the effective action for the massless gauge field in open string theory is an old problem, and has been addressed in many other ways in past literature. Most methods used in the past for calculating the effective vector field action have used world-sheet methods. While the string field theory approach we use here has the advantage that it is a completely off-shell formalism, as just discussed the resulting action has a nonstandard gauge invariance [15]. In world-sheet approaches to this computation, the vector field has the standard gauge transformation rule (2.21) with no further corrections. A general theorem [38] states that there are no deformations of the Yang-Mills gauge invariance which cannot be taken to the usual Yang-Mills gauge invariance by a field redefinition. In accord with this theorem, we identify in this paper field redefinitions which take the massless vector field A_μ in the SFT effective action (2.20) to a gauge field $\hat{A}_\mu$ with the usual gauge invariance. We write the resulting action as

$$\hat{S}[\hat{A}_\mu(x)]\,. \tag{2.22}$$

This action, written in terms of a conventional gauge field, can be compared to previous results on the effective action for the open string massless vector field.

Because the mass-shell condition for the vector field $A_\mu(p)$ in Fourier space is $p^2 = 0$, we can perform a sensible expansion of the action (2.20) as a double expansion in p and A. We write this expansion as

$$S[A_\mu] = \sum_{n=2}^{\infty}\sum_{k=0}^{\infty} S^{[k]}_{A^n} \tag{2.23}$$

where $S^{[k]}_{A^n}$ contains the contribution from all terms of the form $\partial^k A^n$. A similar expansion can be done for $\hat{S}$, and we similarly denote by $\check{S}^{[k]}_{\alpha^m A^n}$ the sum of the terms in $\check{S}$ of the form $\partial^k \alpha^m A^n$.

Because the action $\hat{S}[\hat{A}]$ is a function of a gauge field with conventional gauge transformation rules, this action can be written in a gauge invariant fashion; i.e. in terms of the gauge covariant derivative $\hat{D}_\mu = \partial_\mu - ig_{\rm YM}[\hat{A},\cdot]$ and the field strength $\hat{F}_{\mu\nu}$. For the abelian theory, $\hat{D}_\mu$ is just ∂_μ, and there is a natural double expansion of $\hat{S}$ in terms of p and F. It was shown in [7, 8] that in the abelian theory the set of terms in $\hat{S}$ which depend only on $\hat{F}$, with no additional factors of p (i.e., the terms in $\hat{S}^{[n]}_{\hat{A}^n}$) take the Born-Infeld form (dropping hats)

$$S_{BI} = -\frac{1}{(2\pi g_{YM})^2}\int dx \sqrt{-\det\left(\eta_{\mu\nu} + 2\pi g_{YM} F_{\mu\nu}\right)} \tag{2.24}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.25}$$

is the gauge-invariant field strength. Using $\log(\det M) = \operatorname{tr}(\log(M))$ we can expand in F to get

$$\begin{aligned} S_{BI} = -\frac{1}{(2\pi g_{YM})^2}\int dx \bigg(1 + \frac{(2\pi g_{YM})^2}{4} F_{\mu\nu}F^{\mu\nu} - \frac{(2\pi g_{YM})^4}{8} \times \\ \times \left(F_{\mu\nu}F^{\nu}{}_{\lambda}F^{\lambda}{}_{\sigma}F^{\sigma\mu} - \frac{1}{4}(F_{\mu\nu}F^{\mu\nu})^2\right) + \cdots\bigg)\,. \end{aligned} \tag{2.26}$$

We expect that after the appropriate field redefinition, the result we calculate from string field theory for the effective vector field action (2.20) should contain as a leading part at each power of $\hat{A}$ terms of the form (2.26), as well as higher-derivative terms of the form $\partial^{n+k} A^n$ with $k > 0$. We show in section 5 that this is indeed the case.

The nonabelian theory is more complicated. In the nonabelian theory we must include covariant derivatives, whose commutators mix with field strengths through relations such as

$$[D_\mu, D_\nu]F_{\lambda\sigma} = [F_{\mu\nu}, F_{\lambda\sigma}]\,. \tag{2.27}$$

In this case, there is no systematic double expansion in powers of D and F. It was pointed out by Tseytlin in [40] that when F is taken to be constant, and both commutators $[F, F]$ and covariant derivatives of field strengths DF are taken to be negligible, the nonabelian structure of the theory is irrelevant. In this case, the action reduces to the Born-Infeld form (2.24), where the ordering ambiguity arising from the matrix nature of the field strength F is resolved by the symmetrized trace (STr) prescription whereby all possible orderings of the F's are averaged over. While this observation is correct, it seems that the symmetrized trace formulation of the nonabelian Born-Infeld action misses much of the important physics of the full vector field effective action. In particular, this simplification of the action gives the wrong spectrum around certain background fields, including those which are T-dual to simple intersecting brane configurations [41, 42, 43, 44]. It seems that the only systematic way to deal with the nonabelian vector field action is to include all terms of order F^n at once, counting D at order $F^{1/2}$. The first few terms in the nonabelian vector field action for the bosonic theory were computed in [45, 46, 47]. The terms in the action up to F^4 are given by

$$S_{\text{nonabelian}} = \int -\frac{1}{4}\text{Tr}\, F^2 + \frac{2ig_{\text{YM}}}{3}\,\text{Tr}\left(F^3\right) + \frac{(2\pi g_{YM})^2}{8}\text{STr}\,\left(F^4 - \frac{1}{4}(F^2)^2\right) + \cdots\,. \tag{2.28}$$

In section 6, we show that the effective action we derive from string field theory agrees with (2.28) up to order F^3 after the appropriate field redefinition.

3. Computing the effective action

In this section we develop some tools for calculating low-order terms in the effective action for the massless fields by integrating out all massive fields. Section 3.1 describes a general approach to computing the generating functions for terms in the effective action and gives explicit expressions for the generating functions of cubic and quartic terms. Section 3.2 contains a general derivation of the quartic terms in the effective action for the massless fields. Section 3.3 describes the method we use to numerically approximate the coefficients in the action.

3.1 Generating functions for terms in the effective action

A convenient way of calculating SFT diagrams is to first compute the off-shell amplitude with generic external coherent states

$$|G\rangle = \exp\left(J_{m\mu}a^\mu_{-m} - b_{-m}\mathcal{J}_{bm} + \mathcal{J}_{cm}c_{-m}\right)|p\rangle \tag{3.1}$$

where the index m runs from 1 to ∞ in J^μ_m and $\mathcal{J}_{bm}$ and from 0 to ∞ in $\mathcal{J}_{cm}$.

Let $\Omega_M(p^i, J^i, \mathcal{J}^i_b, \mathcal{J}^i_c;\, 1 \le i \le M)$ be the sum of all connected tree-level diagrams with M external states $|G^i\rangle$. Ω_M is a generating function for all tree-level off-shell M-point amplitudes and can be used to calculate all terms we are interested in in the effective action. Suppose that we are interested in a term in the effective action whose j'th field $\psi^{(j)}_{\mu_1,\dots,\mu_N}(p)$ is associated with the Fock space state

$$\prod_{m,n,q} a^{\mu_m}_{i_m} b_{k_n} c_{l_q} |p\rangle\,. \tag{3.2}$$

We can obtain the associated off-shell amplitude by acting on Ω_M with the corresponding differential operator for each j

$$\int dp\, \psi^{(j)}_{\mu_1,\dots,\mu_N}(p) \prod_{m,n,q} \frac{\partial}{\partial J^j_{i_m \mu_m}} \frac{\partial}{\partial \mathcal{J}^j_{bk_n}} \frac{\partial}{\partial \mathcal{J}^j_{cl_q}} \tag{3.3}$$

and setting J^j, $\mathcal{J}^j_b$, and $\mathcal{J}^j_c$ to 0. Thus, all the terms in the effective action which we are interested in can be obtained from Ω_M.

When we calculate a certain diagram with external states $|G^i\rangle$ by applying formulae (2.16) and (2.17) for inner products of coherent and squeezed states the result has the general form

$$\begin{aligned}\Omega_M = {}& \delta\big(\sum p^r\big) \int \prod_{\ell=1}^{N_{\text{prop}}} d\tau_\ell\, \mathcal{F}(p,\tau)\, \times \\ & \times \exp\left(\frac{1}{2} J^i_m \Delta^{ij}_{mn}(\tau) J^j_n - p^i \Delta^{ij}_{0m}(\tau) J^j_m + p_i \Delta^{ij}_{00}(\tau) p_j + \text{ghosts}\right).\end{aligned} \tag{3.4}$$

A remarkable feature is that (3.4) depends on the sources $J^j, \mathcal{J}^j_b, \mathcal{J}^j_c$ only through the exponent of a quadratic form. Wick's theorem is helpful in writing the derivatives of the exponential in an efficient way. Indeed, the theorem basically reads

$$\prod_{i=1}^{M} \frac{\partial}{\partial J^i_n} \exp\left(\frac{1}{2} J^j_m \Delta^{jk}_{mn} J^k_n\right)\Bigg|_{J^i_n=0} = \text{Sum over all contraction products} \tag{3.5}$$

where the sum is taken over all pairwise contractions, with the contraction between (n,i) and (m,j) carrying the factor Δ^{ij}_{nm}.

Note that Ω_M includes contributions from all the intermediate fields in Feynman-Siegel gauge. To compute the effective action for A_μ we must project out the contribution from intermediate A_μ's.

3.1.1 Three-point generating function

Here we illustrate the idea sketched above with the simple example of the three-point generating function. This generating function provides us with an efficient method of computing the coefficients of the SFT action and the SFT gauge transformation. Plugging $|G^i\rangle$, $1 \le i \le 3$ into the cubic vertex (2.7) and using (2.16), (2.17) to evaluate the inner products we find

$$\Omega_3 = -\frac{\mathcal{N} g}{3}\, \delta\big(\sum_r p^r\big) \exp\left(\frac{1}{2} p^r V^{rs}_{00} p^s - p^r V^{rs}_{0n} J^s_n + \frac{1}{2} J^r_m V^{rs}_{mn} J^s_n - \mathcal{J}^r_{cm} X^{rs}_{mn} \mathcal{J}^s_{bn}\right). \tag{3.6}$$

As an illustration of how this generating function can be used consider the three-tachyon term in the effective action. The external tachyon state is $\int dp\, \phi(p)|p\rangle$. The three-tachyon vertex is obtained from (3.6) by simple integration over momenta and setting the sources to 0. No differentiations are necessary in this case. The three-tachyon term in the action is then

$$-\frac{g}{3}\langle V_3|\, \phi, \phi, \phi\rangle = -\frac{\mathcal{N}g}{3}\int \delta\left(\sum_s p^s\right)\prod_r dp^r\, \phi(p_r)\exp\left(\frac{1}{2}p^r V_{00}^{rs} p^s\right) = -\frac{\mathcal{N}g}{3}\int dx\, \tilde{\phi}(x)^3 \tag{3.7}$$

where

$$\tilde{\phi}(x) = \exp\left(-\frac{1}{2}V_{00}^{11}\partial^2\right)\phi(x)\,. \tag{3.8}$$

For on-shell tachyons, $\partial^2\phi(x) = -\phi(x)$, so that we have

$$-\frac{g}{3}\langle V_3|\, \phi, \phi, \phi\rangle = -\frac{g}{3}\mathcal{N}e^{\frac{3}{2}V_{00}^{11}}\int dx \phi(x)^3 = -\frac{g}{3}\int dx\phi(x)^3\,. \tag{3.9}$$

The normalization constant cancels so that the on-shell three-tachyon amplitude is just $2g$, in agreement with conventions used here and in [48].

3.1.2 Four-point generating function

Now let us consider the generating function for all quartic off-shell amplitudes (see figure 1). The amplitude Ω_4 after contracting all indices can be written as

$$\Omega_4 = \frac{\mathcal{N}^2 g^2}{2}\int_0^\infty d\tau\, e^{\tau(1-(p_1+p_2)^2)}\left\langle \tilde{V}_2\right| |R(1,2)\rangle |R(3,4)\rangle \tag{3.10}$$

where

$$|R(i,j)\rangle^{(k)} = \left\langle G^i\right| \left\langle G^j\right| |V_3\rangle^{(ijk)}\,. \tag{3.11}$$

Applying (2.16), (2.17) to the inner products in (3.11) we get

$$\begin{aligned}|R(1,2)\rangle = \exp\Big(&\frac{1}{2}p_\mu^\alpha U_{00}^{\alpha\beta}p^{\mu\beta} - p_\mu^\alpha U_{0n}^{\alpha\beta}J_n^{\beta\mu} + \frac{1}{2}J_{m\mu}^\alpha U_{mn}^{\alpha\beta}J_n^{\beta\mu} + \\ &+ a_{-m\mu}^{(3)}U_{mn}^{33}a_{-n}^{\mu(3)} + (J_{m\mu}^\alpha U_{mn}^{\alpha 3} - p_\mu^\alpha U_{0n}^{\alpha 3})a_{-n}^{\mu(3)} - \mathcal{J}_{cm}^\alpha X_{mn}^{\alpha\beta}\mathcal{J}_{bn}^\beta + \\ &+ b_{-m}^{(3)}X_{mn}^{3\alpha}\mathcal{J}_{bn}^\alpha - \mathcal{J}_{cm}^\alpha X_{mn}^{\alpha 3}c_{-n}^{(3)} - b_{-m}^{(3)}X_{mn}^{33}c_{-n}^{(3)}\Big)c_0\left|-p^1 - p^2\right\rangle.\end{aligned} \tag{3.12}$$

Here $\alpha, \beta \in 1, 2$ and

$$U^{rs} = \begin{pmatrix} V_{00}^{rs} - V_{00}^{r3} - V_{00}^{3s} + V_{00}^{33} & V_{0n}^{rs} - V_{0n}^{3s} \\ V_{n0}^{rs} - V_{n0}^{r3} & V_{mn}^{rs}\end{pmatrix}. \tag{3.13}$$

Using (2.16), (2.17) one more time to evaluate the inner products in (3.10) we obtain

$$\begin{aligned}\Omega_4 = &\frac{\mathcal{N}^2 g^2}{2}\delta\left(\sum_i p^i\right)\int_0^\infty d\tau e^\tau\, \mathrm{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right)\times \\ &\times\exp\left(\frac{1}{2}p_\mu^i Q_{00}^{ij}p^{j\mu} - p_\mu^i Q_{0n}^{ij}J_n^{j\mu} + \frac{1}{2}J_{m\mu}^i Q_{mn}^{ij}J_n^{j\mu} - \mathcal{J}_{cm}^i \mathcal{Q}_{mn}^{ij}\mathcal{J}_{bn}^j\right).\end{aligned} \tag{3.14}$$

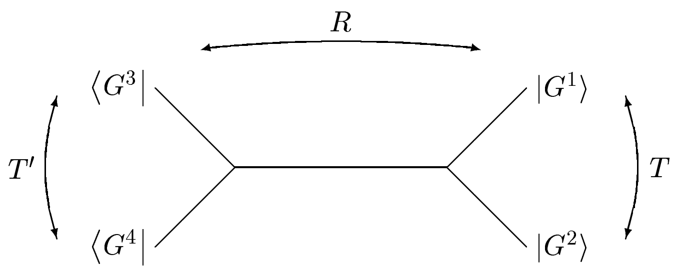

Figure 1: Twists T, T' and reflection R are symmetries of the amplitude.

Here $i, j \in 1, 2, 3, 4$. the matrices $\tilde{V}$ and $\tilde{X}$ are defined by

$$\tilde{V}_{mn} = e^{-\frac{m}{2}\tau} V_{mn} e^{-\frac{n}{2}\tau}\,, \qquad \tilde{X}_{mn} = e^{-\frac{m}{2}\tau} X_{mn} e^{-\frac{n}{2}\tau}\,. \tag{3.15}$$

The matrices Q^{ij} and $\mathcal{Q}^{ij}$ are defined through the tilded matrices $\tilde{Q}^{ij}$ and $\tilde{\mathcal{Q}}^{ij}$

$$\tilde{Q}^{ij}_{mn} = e^{-\frac{m}{2}\tau} Q^{ij}_{mn} e^{-\frac{n}{2}\tau}\,, \qquad \tilde{\mathcal{Q}}^{ij}_{mn} = e^{-\frac{m}{2}\tau} \mathcal{Q}^{ij}_{mn} e^{-\frac{n}{2}\tau} \tag{3.16}$$

where the tilded matrices $\tilde{Q}$ and $\tilde{\mathcal{Q}}$ are defined through $\tilde{V}$, $\tilde{U}$, $\tilde{X}$

$$\begin{aligned} \tilde{Q}^{\alpha\beta} &= \tilde{U}^{\alpha 3}\frac{1}{1-\tilde{V}^2}\tilde{V}\tilde{U}^{3\beta} + \tilde{U}^{\alpha\beta}\,, & \tilde{\mathcal{Q}}^{\alpha\beta} &= \tilde{X}^{\alpha 3}\frac{1}{1-\tilde{X}^2}\tilde{X}\tilde{X}^{3\beta} + \tilde{X}^{\alpha\beta}\,, \\ \tilde{Q}^{\alpha\alpha'}_{mn} &= -\left(\tilde{U}^{\alpha 3}\frac{1}{1-\tilde{V}^2}C\tilde{U}^{3\alpha'}\right)_{mn} + \delta_{0m}\delta_{0n}\tau\,, & \tilde{\mathcal{Q}}^{\alpha\alpha'} &= -\tilde{X}^{\alpha 3}\frac{1}{1-\tilde{X}^2}C\tilde{X}^{3\alpha'} \end{aligned} \tag{3.17}$$

with $\alpha, \beta \in 1, 2$; $\alpha', \beta' \in 3, 4$. The matrix $\tilde{U}$ includes zero modes while $\tilde{V}$ does not, so one has to understand $\tilde{U}\tilde{V}$ in (3.17) as a product of $\tilde{U}$, where the first column is dropped, and $\tilde{V}$. Similarly $\tilde{V}\tilde{U}$ is the product of $\tilde{V}$ and $\tilde{U}$ with the first row of $\tilde{U}$ omitted.

The matrices Q^{ij} are not all independent for different i and j. The four-point amplitude is invariant under the twist transformation of either of the two vertices as well as under the interchange of the two (see figure 1). In addition the whole block matrix Q^{ij}_{mn} has been defined in such a way that it is symmetric under the simultaneous exchange of i with j and m with n. Algebraically, we can use properties (A.7a), (A.7b), (A.7c) of Neumann coefficients to show that the matrices Q^{ij} satisfy

$$\begin{aligned} (Q^{\alpha\beta})^T &= Q^{\beta\alpha}\,, & CQ^{\alpha\beta}C &= Q^{3-\alpha\, 3-\beta}\,, & Q^{\alpha\beta} &= Q^{\alpha+2\,\beta+2}\,, \\ (Q^{\alpha'\beta'})^T &= Q^{\beta'\alpha'}\,, & CQ^{\alpha'\beta'}C &= Q^{7-\alpha'\, 7-\beta'}\,, & Q^{\alpha'\beta'} &= Q^{\alpha'-2\,\beta'-2}\,, \\ (Q^{\alpha\alpha'})^T &= Q^{\alpha'\alpha}\,, & CQ^{\alpha\alpha'}C &= Q^{3-\alpha\, 7-\alpha'}\,, & Q^{\alpha\alpha'} &= Q^{\alpha+2\,\alpha'-2}\,. \end{aligned} \tag{3.18}$$

The analogous relations are satisfied by ghost matrices $\mathcal{Q}$.

Note that we still have some freedom in the definition of the zero modes of the matter matrices Q. Due to the momentum conserving delta function we can add to the exponent in the integrand of (3.14) any expression proportional to $\sum p_i$. To fix this freedom we require that after the addition of such a term the new matrices $\bar{Q}$ satisfy $\bar{Q}^{ii}_{00} = \bar{Q}^{ii}_{0n} = 0$. This gives

$$\bar{Q}^{ij}_{00} = Q^{ij}_{00} - \frac{1}{2}Q^{jj}_{00} - \frac{1}{2}Q^{ii}_{00}\,, \qquad \bar{Q}^{ij}_{0n} = Q^{ij}_{0n} - Q^{jj}_{0n}\,. \tag{3.19}$$

and $\bar{Q}^{ij}_{mn} = Q^{ij}_{mn}$ for $m,n > 0$. The addition of any term proportional to $\sum p_i$ corresponds in coordinate space to the addition of a total derivative. In coordinate space we have essentially integrated by parts the terms $\partial_\sigma\partial^\sigma\psi_{\mu_1\cdots\mu_n}(x)$ and $\partial^{\mu_j}\psi_{\mu_1\cdots\mu_j\cdots\mu_n}(x)$ thus fixing the freedom of integration by parts.

To summarize, we have rewritten Ω_4 in terms of $\bar{Q}$'s as

$$\begin{aligned}\Omega_4 = {} & \frac{\mathcal{N}^2 g^2}{2}\delta\left(\sum_i p^i\right)\int_0^\infty d\tau e^\tau \,\mathrm{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right)\times \\ & \times\exp\left(\frac{1}{2}p^i_\mu \bar{Q}^{ij}_{00}p^{j\mu} - p^i_\mu \bar{Q}^{ij}_{0n}J^{j\mu}_n + \frac{1}{2}J^i_{m\mu}\bar{Q}^{ij}_{mn}J^{j\mu}_n - \mathcal{J}^i_{cm}\mathcal{Q}^{ij}_{mn}\mathcal{J}^j_{bn}\right).\end{aligned} \tag{3.20}$$

There are only three independent matrices $\bar{Q}$. For later use we find it convenient to denote the independent $\bar{Q}$'s by $A = \bar{Q}^{12}$, $B = \bar{Q}^{13}$, $C = \bar{Q}^{14}$. Then the matrix $\bar{Q}^{ij}_{mn}$ can be written as

$$\bar{Q}^{ij}_{mn} = \begin{pmatrix} 0 & A_{mn} & B_{mn} & C_{mn} \\ (-1)^{m+n}A_{mn} & 0 & (-1)^{m+n}C_{mn} & (-1)^{m+n}B_{mn} \\ B_{mn} & C_{mn} & 0 & A_{mn} \\ (-1)^{m+n}C_{mn} & (-1)^{m+n}B_{mn} & (-1)^{m+n}A_{mn} & 0 \end{pmatrix}. \tag{3.21}$$

In the next section we derive off-shell amplitudes for the massless fields by differentiating Ω_4. The generating function Ω_4 defined in (3.20) and supplemented with the definition of the matrices $\tilde{V}$, $\tilde{X}$, $\bar{Q}$, $\mathcal{Q}$ given in (3.13), (3.15), (3.16), (3.17), (3.19) and (3.21) provides us with all information about the four-point tree-level off-shell amplitudes.

3.2 Effective action for massless fields

In this subsection we compute explicit expressions for the general quartic off-shell amplitudes of the massless fields, including derivatives to all orders. Our notation for the massless fields is, as in (2.2),

$$|\Phi_{\mathrm{massless}}\rangle = \int d^dp\,\left(A_\mu(p)a^\mu_{-1} - i\alpha(p)b_{-1}c_0\right)|p\rangle\,. \tag{3.22}$$

External states with A_μ and α in the k'th Fock space are inserted using

$$D^{A,k}_\mu = \int dp A_\mu(p)\frac{\partial}{\partial J^k_{1\mu}}\bigg|_{J^k=\mathcal{J}^k_{b,c}=0} \tag{3.23}$$

and

$$D^{\alpha,k} = -i \int dp\, \alpha(p) \frac{\partial}{\partial \mathcal{J}^k_{b1}} \frac{\partial}{\partial \mathcal{J}^k_{c0}} \bigg|_{J^k = \mathcal{J}^k_{b,c} = 0} . \tag{3.24}$$

We can compute all quartic terms in the effective action $\check{S}[A_\mu, \alpha]$ by computing quartic off-shell amplitudes for the massless fields by acting on Ω_4 with D^A and D^α. First consider the quartic term with four external A's. The relevant off-shell amplitude is given by $\prod_{i=1}^4 D^{A,i}_{\mu_i} \Omega_4$ where Ω_4 is given in (3.20) and $D^{A,i}_{\mu_i}$ is given in (3.23). Performing the differentiations we get

$$\begin{aligned} S_{A^4} = & \frac{1}{2} \mathcal{N}^2 g^2 \int \prod_i dp^i \delta\left(p^1 + p^2 + p^3 + p^4\right) A^{\mu_1}(p_1) A^{\mu_2}(p_2) A^{\mu_3}(p_3) A^{\mu_4}(p_4) \times \\ & \times \int_0^\infty d\tau e^\tau \operatorname{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right) \left(\mathcal{I}^0_{A^4} + \mathcal{I}^2_{A^4} + \mathcal{I}^4_{A^4}\right) \exp\left(\frac{1}{2} p^i_\mu \bar{Q}^{ij}_{00} p^{j\mu}\right) . \end{aligned} \tag{3.25}$$

Here $\mathcal{I}^0_{A^4}$, $\mathcal{I}^2_{A^4}$, $\mathcal{I}^4_{A^4}$ are defined by

$$\begin{aligned} \mathcal{I}^0_{A^4} = & \frac{1}{8} \sum_{i_i \neq i_j} \bar{Q}^{i_1 i_2}_{11} \bar{Q}^{i_3 i_4}_{11} \eta_{\mu_{i_1} \mu_{i_2}} \eta_{\mu_{i_3} \mu_{i_4}} , \\ \mathcal{I}^2_{A^4} = & \frac{1}{4} \sum_{i_i \neq i_j} \bar{Q}^{i_1 i_2}_{11} \bar{Q}^{i_3 j_1}_{10} \bar{Q}^{i_4 j_2}_{10} p^{j_1}_{\mu_{i_3}} p^{j_2}_{\mu_{i_4}} \eta_{\mu_{i_1} \mu_{i_2}} , \\ \mathcal{I}^4_{A^4} = & \bar{Q}^{1i}_{10} \bar{Q}^{2j}_{10} \bar{Q}^{3k}_{10} \bar{Q}^{4l}_{10} p^i_{\mu_1} p^j_{\mu_2} p^k_{\mu_3} p^l_{\mu_4} . \end{aligned} \tag{3.26}$$

Other amplitudes with α's and A's all have the same pattern as (3.25). The amplitude with one α and three A's is obtained by replacing $A_{\mu_{i_1}}(p^{i_1})$ in formula (3.25) with $i\alpha(p^{i_1})$ and the sum of $\mathcal{I}^{0,2,4}_{A^4}$ with the sum of

$$\begin{aligned} \mathcal{I}^1_{\alpha A^3} &= \frac{1}{2} \sum_{i_i \neq i_j} \mathcal{Q}^{i_1 i_1}_{01} \bar{Q}^{i_2 i_3}_{11} \bar{Q}^{i_4 k}_{10} p^k_{\mu_{i_4}} \eta_{\mu_{i_2} \mu_{i_3}} , \\ \mathcal{I}^3_{\alpha A^3} &= \frac{1}{6} \sum_{i_i \neq i_j} \mathcal{Q}^{i_1 i_1}_{01} \bar{Q}^{i_2 j}_{10} \bar{Q}^{i_3 k}_{10} \bar{Q}^{i_4 l}_{10} p^j_{\mu_{i_2}} p^k_{\mu_3} p^l_{\mu_4} . \end{aligned} \tag{3.27}$$

The amplitude with two A's and two α's is obtained by replacing $A_{\mu_{i_1}}(p^{i_1}) A_{\mu_{i_2}}(p^{i_2})$ with $-\alpha(p^{i_1})\alpha(p^{i_2})$ and the sum of $\mathcal{I}^{0,2,4}_{A^4}$ with the sum of

$$\begin{aligned} \mathcal{I}^0_{\alpha^2 A^2} &= \frac{1}{4} \sum_{i_i \neq i_j} (\mathcal{Q}^{i_1 i_1}_{01} \mathcal{Q}^{i_2 i_2}_{01} - \mathcal{Q}^{i_1 i_2}_{01} \mathcal{Q}^{i_2 i_1}_{01}) \bar{Q}^{i_3 i_4}_{11} \eta_{\mu_{i_3} \mu_{i_4}} , \\ \mathcal{I}^2_{\alpha^2 A^2} &= \frac{1}{4} \sum_{i_i \neq i_j} (\mathcal{Q}^{i_1 i_1}_{01} \mathcal{Q}^{i_2 i_2}_{01} - \mathcal{Q}^{i_1 i_2}_{01} \mathcal{Q}^{i_2 i_1}_{01}) \bar{Q}^{i_3 k}_{10} \bar{Q}^{i_4 l}_{10} p^k_{\mu_3} p^l_{\mu_4} . \end{aligned} \tag{3.28}$$

It is straightforward to write down the analogous expressions for the terms of order $\alpha^3 A$ and α^4. However, as we shall see later, it is possible to extract all the information about the coefficients in the expansion of the effective action for A_μ in powers of field strength up to F^4 from the terms of order $A^4, A^3\alpha$, and $A^2\alpha^2$.

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The off-shell amplitudes (3.25), (3.26), (3.27) and (3.28) include contributions from the intermediate gauge field. To compute the quartic terms in the effective action we must subtract, if nonzero, the amplitude with intermediate A_μ. In the case of the abelian theory this amplitude vanishes due to the twist symmetry. In the nonabelian case, however, the amplitude with intermediate A_μ is nonzero. The level truncation method in the next section makes it easy to subtract this contribution at the stage of numerical computation.

As in (2.23), we expand the effective action in powers of p. As an example of a particular term appearing in this expansion, let us consider the space-time independent (zero-derivative) term of (3.25). In the abelian case there is only one such term: $A_\mu A^\mu A_\nu A^\nu$. The coefficient of this term is

$$\gamma = \frac{1}{2}\mathcal{N}^2 g^2 \int_0^\infty d\tau e^\tau \operatorname{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right)(A_{11}^2 + B_{11}^2 + C_{11}^2) \tag{3.29}$$

where the matrices A, B and C are those in (3.21). In the nonabelian case there are two terms, $\operatorname{Tr}(A_\mu A^\mu A_\nu A^\nu)$ and $\operatorname{Tr}(A_\mu A_\nu A^\mu A^\nu)$, which differ in the order of gauge fields. The coefficients of these terms are obtained by keeping $A_{11}^2 + C_{11}^2$ and B_{11}^2 terms in (3.29) respectively.

3.3 Level truncation

Formula (3.29) and analogous formulae for the coefficients of other terms in the effective action contain integrals over complicated functions of infinite-dimensional matrices. Even after truncating the matrices to finite size, these integrals are rather difficult to compute. To get numerical values for the terms in the effective action, we need a good method for approximately evaluating integrals of the form (3.29). In this subsection we describe the method we use to approximate these integrals. For the four-point functions, which are the main focus of the computations in this paper, the method we use is equivalent to truncating the summation over intermediate fields at finite field level. Because the computation is carried out in the oscillator formalism, however, the complexity of the computation only grows polynomially in the field level cutoff.

Tree diagrams with four external fields have a single internal propagator with Schwinger parameter τ. It is convenient to do a change of variables

$$\sigma = e^{-\tau}\,. \tag{3.30}$$

We then truncate all matrices to size $L \times L$ and expand the integrand in powers of σ up to σ^{M-2}, dropping all terms of higher order in σ. We denote this approximation scheme by $\{L, M\}$. The σ^n term of the series contains the contribution from all intermediate fields at level $k = n+2$, so in this approximation scheme we are keeping all oscillators $a^\mu_{k\leq L}$ in the string field expansion, and all intermediate particles in the diagram of mass $m^2 \leq M-1$. We will use the approximation scheme $\{L, L\}$ throughout this paper. This approximation really imposes only one restriction— the limit on the mass of the intermediate particle. It is perhaps useful to compare the approximation scheme we are using here with those used in previous work on related problems. In [21] analogous integrals were computed by numerical integration. This corresponds to $\{L, \infty\}$ truncation. In earlier papers on level truncation in string field theory, such as [49, 50, 51] and many others, the (L, M) truncation scheme was used, in which fields of mass up to $L-1$ and interaction vertices with total mass of fields in the vertex up to $M-3$ are kept. Our $\{L, L\}$ truncation scheme is equivalent to the $(L, L+2)$ truncation scheme by that definition.

To explicitly see how the σ expansion works let us write the expansion in σ of a generic integrand and take the integral term by term

$$\int_0^1 \frac{d\sigma}{\sigma^2} \sigma^{p^2} \sum_{n=0}^{\infty} c_n(p^i) \sigma^n = \sum_{n=0}^{\infty} \frac{c_n(p^i)}{p^2 + n - 1} . \tag{3.31}$$

Here $p = p_1 + p_2 = p_3 + p_4$ is the intermediate momentum. This is the expansion of the amplitude into poles corresponding to the contributions of (open string) intermediate particles of fixed level. We can clearly see that dropping higher powers of σ in the expansion means dropping the contribution of very massive particles. We also see that to subtract the contribution from the intermediate fields A_μ and α we can simply omit the term $c_1(p)\sigma^{p^2-1}$ in (3.31).

While the Taylor expansion of the integrand might seem difficult, it is in fact quite straightforward. We notice that $\tilde{V}^{rs}$, and $\tilde{X}^{rs}$ are both of order σ. Therefore we can simply expand the integrand in powers of matrices $\tilde{V}$ and $\tilde{X}$. For example, the determinant of the matter Neumann coefficients is

$$\mathrm{Det}(1 - \tilde{V}^2)^{-13} = \exp\left(-13 \,\mathrm{Tr}\,\mathrm{Log}(1 - \tilde{V}^2)\right) . \tag{3.32}$$

Looking again at (3.25) we notice that the only matrix series' that we will need are $\mathrm{Log}(1 - \tilde{V}^2)$ for the determinant (and the analogue for $\tilde{X}$) and $1/(1 - \tilde{V}^2)$ for $\bar{Q}^{ij}$. Computation of these series is straightforward.

It is also easy to estimate how computation time grows with L and M. The most time consuming part of the Taylor expansion in σ is the matrix multiplication. Recall that $\tilde{V}$ is an $L \times L$ matrix whose coefficients are proportional to σ^n at leading order. Elements of $\tilde{V}^k$ are polynomials in σ with M terms. To construct a series $a_0 + a_1\tilde{V} + \cdots + a_M\tilde{V}^M + \mathcal{O}(\sigma^{M+1})$ we need M matrix multiplications $\tilde{V}^k \cdot \tilde{V}$. Each matrix multiplication consists of L^3 multiplications of its elements. Each multiplication of the elements has on the average $M/2$ multiplications of monomials. The total complexity therefore grows as L^3M^2.

The method just described allows us to compute approximate coefficients in the effective action at any particular finite level of truncation. In [21], it was found empirically that the level truncation calculation gives approximate results for finite on-shell and off-shell amplitudes with errors which go as a power series in $1/L$. Based on this observation, we can perform a least-squares fit on a finite set of level truncation data for a particular term in the effective action to attain a highly accurate estimate of the coefficient of that term. We use this method to compute coefficients of terms in the effective action which are quartic in A throughout the remainder of this paper.

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4. The Yang-Mills action

In this section we assemble the Yang-Mills action, picking the appropriate terms from the two, three and four-point Green functions. We write the Yang-Mills action as

$$S_{YM} = \int d^d x \mathrm{Tr}\Big(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu + \\ + i g_{YM}\partial_\mu A_\nu [A^\mu, A^\nu] + \frac{1}{4} g_{YM}^2 [A_\mu, A_\nu][A^\mu, A^\nu]\Big) . \tag{4.1}$$

In section 4.1 we consider the quadratic terms of the Yang-Mills action. In section 4.2 we consider the cubic terms and identify the Yang-Mills coupling constant g_{YM} in terms of the SFT (three tachyon) coupling constant g. This provides us with the expected value for the quartic term. In section 4.3 we present the results of a numerical calculation of the (space-time independent) quartic terms and verify that we indeed get the Yang-Mills action.

4.1 Quadratic terms

The quadratic term in the action for massless fields, calculated from (2.4), and (2.6) is

$$\check{S}_{A^2} = \int d^d x \, \mathrm{Tr}\Big(-\frac{1}{2}\partial_\mu A_\nu \, \partial^\mu A^\nu - \alpha^2 + \sqrt{2}\alpha \, \partial_\mu A^\mu \Big) . \tag{4.2}$$

Completing the square in α and integrating the term $(\partial A)^2$ by parts we obtain

$$\check{S}_{A_2} = \int d^d x \mathrm{Tr}\Big(-\frac{1}{2}\partial_\mu A_\nu \, \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \, \partial^\nu A^\mu - B^2 \Big) \tag{4.3}$$

where we denote

$$B = \alpha - \frac{1}{\sqrt{2}}\partial_\mu A^\mu . \tag{4.4}$$

Eliminating α using the leading-order equation of motion, $B = 0$, leads to the quadratic terms in (4.1). Subleading terms in the equation of motion for α lead to higher-order terms in the effective action, to which we return in the following sections.

4.2 Cubic terms

The cubic terms in the action for the massless fields are obtained by differentiating (3.6). The terms cubic in A are given by

$$\check{S}_{A^3} = \frac{\mathcal{N} g}{3} \int \prod_i dp_i \delta \left(\sum_j p_j \right) \mathrm{Tr}\Big(A_\mu(p_1) A_\nu(p_2) A_\lambda(p_3) \Big) \exp\left(\frac{1}{2} p^r V_{00}^{rr} p^r \right) \times \\ \times \Big(\big(\eta^{\nu\lambda} p^{r\mu} V_{01}^{r1} V_{11}^{32} + \eta^{\mu\lambda} p^{r\nu} V_{01}^{r2} V_{11}^{13} + \eta^{\mu\nu} p^{r\lambda} V_{01}^{r3} V_{11}^{12} \big) + p^{r\mu} V_{01}^{r1} p^{s\nu} V_{01}^{s2} p^{t\lambda} V_{01}^{t3} \Big) . \tag{4.5}$$

To compare with the Yang-Mills action we perform a Fourier transform and use the properties of the Neumann coefficients to combine similar terms. We then get

$$\begin{aligned}\check{S}_{A^3} = -i\mathcal{N}g\int dx\,\mathrm{Tr}\Big(& V_{11}^{12}V_{01}^{12}\big(\partial_\mu\tilde{A}_\nu[\tilde{A}^\mu,\tilde{A}^\nu]\big) + \\ & + \frac{1}{3}(V_{01}^{12})^3\big(\partial_\lambda\tilde{A}^\mu\partial_\mu\tilde{A}^\nu\partial_\nu\tilde{A}^\lambda - \partial_\nu\tilde{A}^\mu\partial_\lambda\tilde{A}^\nu\partial_\mu\tilde{A}^\lambda\big) + \\ & + \big(V_{01}^{12}\big)^3[\tilde{A}_\nu,\partial^\lambda\tilde{A}_\mu]\partial^\mu\partial^\nu\tilde{A}_\lambda\Big)\end{aligned} \tag{4.6}$$

where, following the notation introduced in (3.8), we have

$$\tilde{A}_\mu = \exp\left(-\frac{1}{2}V_{00}^{11}\partial^2\right)A_\mu\,. \tag{4.7}$$

To reproduce the cubic terms in the Yang-Mills action, we are interested in the terms in (4.6) of order ∂A^3. The remaining terms and the terms coming from the expansion of the exponential of derivatives contribute to higher-order terms in the effective action, which we discuss later. The cubic terms in the action involving the α field are

$$\begin{aligned}&\check{S}_{A\alpha^2} = -i\mathcal{N}gV_{01}^{12}\left(X_{01}^{12}\right)^2\int dx\,\mathrm{Tr}\big(\tilde{A}^\mu\left[\partial_\mu\tilde{\alpha},\tilde{\alpha}\right]\big)\,,\\ &\check{S}_{A^2\alpha} = \check{S}_{\alpha^3} = 0\,.\end{aligned} \tag{4.8}$$

$\check{S}_{A^2\alpha}$ vanishes because $X_{01}^{11}=0$, and $\check{S}_{\alpha^3}$ is zero because $[\alpha,\alpha]=0$. After α is eliminated using its equation of motion, (4.8) first contributes terms at order ∂^3A^3.

The first line of (4.6) contributes to the cubic piece of the F^2 term. Substituting the explicit values of the Neumann coefficients:

$$\begin{aligned}V_{00}^{11} &= -\log\left(\frac{27}{16}\right), & V_{11}^{12} &= \frac{16}{27}\,,\\ V_{01}^{12} &= -\frac{2\sqrt{2}}{3\sqrt{3}}\,, & X_{01}^{12} &= \frac{4}{(3\sqrt{3})}\,.\end{aligned} \tag{4.9}$$

we write the lowest-derivative term of (4.6) as

$$S_{A^3}^{[1]} = i\frac{g}{\sqrt{2}}\int d^dx\,\mathrm{Tr}\left(\partial_\mu A_\nu[A^\mu,A^\nu]\right). \tag{4.10}$$

We can now predict the value of the quartic amplitude at zero momentum. From (4.1) and (4.10) we see that the Yang-Mills constant is related to the SFT coupling constant by

$$g_{YM} = \frac{1}{\sqrt{2}}g\,. \tag{4.11}$$

This is the same relation between the gauge boson and tachyon couplings as the one given in formula (6.5.14) of Polchinski [48]. We expect the nonderivative part of the quartic term in the effective action to add to the quadratic and cubic terms to form the full Yang-Mills action, so that

$$S_{A_4}^{[0]} = \frac{1}{4}g_{YM}^2[A_\mu,A_\nu]^2\,. \tag{4.12}$$

4.3 Quartic terms

As we have just seen, to get the full Yang-Mills action the quartic terms in the effective action at $p = 0$ must take the form (4.12). We write the nonderivative part of the SFT quartic effective action as

$$S^{[0]}_{A^4} = g^2 \int dx \left(\gamma_+ \mathrm{Tr}(A_\mu A^\mu)^2 + \frac{1}{4}\gamma_- \mathrm{Tr}[A_\mu, A^\nu]^2 \right). \tag{4.13}$$

We can use the method described in section 3.3 to numerically approximate the coefficients γ_+ and γ_- in level truncation. In the limit $L \to \infty$ we expect that $\gamma_+ \to 0$ and that $\gamma_- \to g_{YM}^2/g^2 = 1/2$. As follows from formula (3.29) and the comment below it $\gamma_\pm$ are given by:

$$\begin{aligned}
\gamma_+ &= \frac{1}{2}\mathcal{N}^2 \int_0^\infty e^\tau d\tau\, \mathrm{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right)(A_{11}^2 + B_{11}^2 + C_{11}^2)\,, \\
\gamma_- &= \mathcal{N}^2 \int_0^\infty e^\tau d\tau\, \mathrm{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right) B_{11}^2\,.
\end{aligned} \tag{4.14}$$

We have calculated these integrals including contributions from the first 100 levels. We have found that as the level L increases the coefficients γ_+ and γ_- indeed converge to their expected values.[2] The leading term in the deviation decays as $1/L$ as expected. Figure 2 shows the graphs of $\gamma_\pm(L)$ vs L.

Table 1 explicitly lists the results from the first 10 levels. At level 100 we get $\gamma_+ = 0.0037$, $\gamma_- = 0.4992$ which is within 0.5% of the expected values. One can improve precision even more by doing a least-squares fit of $\gamma_\pm(L)$ with an expansion in powers of $1/L$ with indeterminate coefficients. The contributions to $\gamma_\pm$ from the even and odd level fields are oscillatory. Thus, the fit for only even or only odd levels works much better. The least-squares fit for the last 25 even levels gives

Level	$\gamma_+(n)$	$\gamma_-(n)$	$\gamma_-(n) - \frac{1}{2}$
0	−0.844	0	−0.500
2	−0.200	0.592	0.092
3	−0.200	0.417	−0.083
4	−0.097	0.504	0.004
5	−0.097	0.468	−0.032
6	−0.063	0.495	−0.005
7	−0.063	0.483	−0.017
8	−0.047	0.494	−0.006
9	−0.047	0.487	−0.013
10	−0.037	0.494	−0.006

Table 1: Coefficients of the constant quartic terms in the action for the first 10 levels.

$$\begin{aligned}
\gamma_+(L) &\approx -5\cdot 10^{-8} - \frac{0.35807}{L} - \frac{0.0091}{L^2} - \frac{1.6}{L^3} + \frac{15}{L^4} + \cdots \\
\gamma_-(L) &\approx \frac{1}{2} - 2\cdot 10^{-8} - \frac{0.0795838}{L} + \frac{0.1212}{L^2} + \frac{1.02}{L^3} - \frac{1.24}{L^4} + \cdots\,.
\end{aligned} \tag{4.15}$$

[2] We were recently informed of an analytic proof of this result in SFT, which will appear in [52]

[3] Note that in [53], an earlier attempt was made to calculate the coefficients $\gamma_\pm$ from SFT. The results in that paper are incorrect; the error made there was that odd-level fields, which do not contribute in the abelian action due to twist symmetry, were neglected. As these fields do contribute in the nonabelian theory, the result for γ_- obtained in [53] had the wrong numerical value. Our calculation here automatically includes odd-level fields, and reproduces correctly the expected value.

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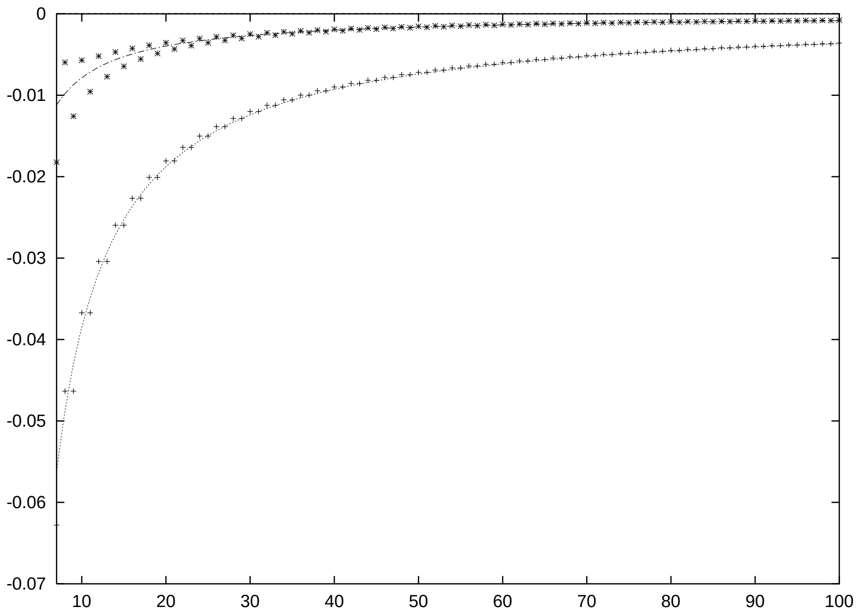

Figure 2: Deviation of the coefficients of quartic terms in the effective action from the expected values, as a function of the level of truncation L. The coefficient γ_+ is shown with crosses and $\gamma_- - 1/2$ is shown with stars. The curves given by fitting with a power series in $1/L$ are graphed in both cases.

We see that when $L \to \infty$ the fitted values of $\gamma_\pm$ are in agreement with the Yang-Mills quartic term to 7 digits of precision.[3]

The calculations we have described so far provide convincing evidence that the SFT effective action for A_μ reproduces the nonabelian Yang-Mills action. This is encouraging in several respects. First, it shows that our method of computing Feynman diagrams in SFT is working well. Second, the agreement with on-shell calculations is another direct confirmation that cubic SFT provides a correct off-shell generalization of bosonic string theory. Third, it encourages us to extend these calculations further to get more information about the full effective action of A_μ.

5. The abelian Born-Infeld action

In this section we consider the abelian theory, and compute terms in the effective action which go beyond the leading Yang-Mills action computed in the previous section. As discussed in section 2.3, we expect that the effective vector field theory computed from string field theory should be equivalent under a field redefinition to a theory whose leading terms at each order in A take the Born-Infeld form (2.26). In this section we give evidence that this is indeed the case. In the abelian theory, the terms at order A^3 vanish identically, so the quartic terms are the first ones of interest beyond the quadratic Yang-Mills action. In subsection 5.1 we use our results on the general quartic term from 3.2 to explicitly compute the terms in the effective action at order $\partial^2 A^4$. We find that these terms are nonvanishing. We find, however, that the gauge invariance of the effective action constrains the terms at this order to live on a one-parameter family of terms related through field redefinitions, and that the terms we find are generated from the Yang-Mills terms $\hat{F}^2$ with an appropriate field redefinition. We discuss general issues of field redefinition and gauge invariance in subsection 5.2; this discussion gives us a framework with which to analyze more complicated terms in the effective action. In subsection 5.3 we analyze terms of the form $\partial^4 A^4$, and show that these terms indeed take the form predicted by the Born-Infeld action after the appropriate field redefinition. In subsection 5.4 we consider higher-order terms with no derivatives, and give evidence that terms of order $(A \cdot A)^n$ vanish up to $n = 5$ in the string field theory effective action.

5.1 Terms of the form $\partial^2 A^4$

In the abelian theory, all terms in the Born-Infeld action have the same number of fields and derivatives. If we assume that the effective action for A_μ calculated in SFT directly matches the Born-Infeld action (plus higher-order derivative corrections) we would expect the $\partial^2 A^4$ terms in the expansion of the effective action to vanish. The most general form of the quartic terms with two derivatives is parameterized as[4]

$$S^2_{A^4} = g^2 \int d^{26}x \Big(c_1 A_\mu A^\mu \partial_\sigma A_\nu \partial^\sigma A^\nu + c_2 A_\mu A_\nu \partial_\sigma A^\mu \partial^\sigma A^\nu + c_3 A_\mu A_\nu \partial^\mu A_\sigma \partial^\nu A^\sigma + \\ + c_4 A_\mu A_\nu \partial^\nu A_\sigma \partial^\sigma A^\mu + c_5 A_\mu A_\nu A_\sigma \partial^\mu \partial^\nu A^\sigma + c_6 A_\mu A^\mu \partial_\sigma A^\nu \partial_\nu A^\sigma \Big) . \quad (5.1)$$

When α is eliminated from the massless effective action $\check{S}$ using the equation of motion, we might then expect that all coefficients c_n in the resulting action (5.1) should vanish. Let us now compute these terms explicitly. From (4.3) and (4.8) we see that the equation of motion for α in the effective theory of the massless fields reads (in the abelian theory)

$$\alpha = \frac{1}{\sqrt{2}} \partial^\mu A_\mu + \mathcal{O}((A, \alpha)^3) . \quad (5.2)$$

The coefficients $c_1, \ldots, c_6$ thus get contributions from the two-derivative term of (3.25), the one-derivative term of (3.27) and the zero-derivative term of (3.28). We first consider the contribution from the four-gauge boson amplitude (3.25). All the expressions for these contributions, which we denote $(\delta c_i)_{A^4}$, are of the form

$$(\delta c_i)_{A^4} = \frac{1}{2} \mathcal{N}^2 \int_0^\infty d\tau e^\tau \,\mathrm{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) P_{\partial^2 A^4, i}(A, B, C) . \quad (5.3)$$

Here $P_{\partial^2 A^4, i}$ are polynomials in the elements of the matrices A, B and C which were defined in (3.21). It is straightforward to derive expressions for the polynomials $P_{\partial^2 A^4, i}$ from (3.25)

[4] Recall that in section 3.1.2 we fixed the integration by parts freedom by integrating by parts all terms with $\partial^2 A_\lambda$ and $\partial \cdot A$. Formula (5.1) gives the most general combination of terms with four A's and two derivatives that do not have $\partial^2 A_\lambda$ and $\partial \cdot A$.

and (3.26), so we just give the result here

$$\begin{aligned}
P_{\partial^2 A^4,1} &= -2\big(A_{11}^2 A_{00} + B_{11}^2 B_{00} + C_{11}^2 C_{00}\big)\,,\\
P_{\partial^2 A^4,2} &= -2\big(A_{11}^2(B_{00}+C_{00}) + B_{11}^2(A_{00}+C_{00}) + C_{11}^2(A_{00}+B_{00})\big)\,,\\
P_{\partial^2 A^4,3} &= 2\big(A_{11}(B_{10}^2+C_{10}^2) - B_{11}(A_{10}^2+C_{10}^2) + C_{11}(A_{10}^2+B_{10}^2)\big)\,,\\
P_{\partial^2 A^4,4} &= 4\big(A_{11}A_{10}(B_{10}+C_{10}) - B_{11}B_{10}(A_{10}+C_{10}) + C_{11}C_{10}(A_{10}+B_{10})\big)\,,\\
P_{\partial^2 A^4,5} &= 4\big(A_{11}B_{10}C_{10} - B_{11}A_{10}C_{10} + C_{11}A_{10}B_{10}\big)\,,\\
P_{\partial^2 A^4,6} &= 2\big(A_{11}A_{10}^2 - B_{11}B_{10}^2 + C_{11}C_{10}^2\big)\,.
\end{aligned} \tag{5.4}$$

The terms in the effective action $\check{S}$ which contain α's and contribute to $S[A]$ at order $\partial^2 A^4$ can similarly be computed from (3.27) and (3.28) and are given by

$$\check{S}^{[1]}_{\alpha A^3} + \check{S}^{[0]}_{\alpha^2 A^2} = g^2 \int d^{26}x \big(\sigma_1 \alpha A_\mu A_\nu \partial^\mu A^\nu + \sigma_2 \partial^\mu \alpha A_\mu A_\nu A^\nu + \sigma_3 \alpha^2 A_\mu A^\mu\big) \tag{5.5}$$

where the coefficients σ_i are given by

$$\begin{aligned}
P_{\partial\alpha A^3,1} &= 4\mathcal{Q}^{11}_{01}\big(A_{11}(B_{10}+C_{10}) - B_{11}(A_{10}+C_{10}) + C_{11}(B_{10}+A_{10})\big)\,,\\
P_{\partial\alpha A^3,2} &= 4\mathcal{Q}^{11}_{01}\big(A_{11}A_{10} - B_{11}B_{10} + C_{11}C_{10}\big)\,,\\
P_{\alpha^2 A^2} &= 2\big((\mathcal{Q}^{11}_{01})^2 - (\mathcal{Q}^{12}_{01})^2\big)A_{11} - \big((\mathcal{Q}^{11}_{01})^2 - (\mathcal{Q}^{13}_{01})^2\big)B_{11}\big((\mathcal{Q}^{11}_{01})^2 - (\mathcal{Q}^{14}_{01})^2\big)C_{11}\,.
\end{aligned} \tag{5.6}$$

Computation of the integrals up to level 100 and using a least-squares fit gives us

$$\begin{aligned}
&(\delta c_1)_{A^4} \approx -2.1513026\,, && (\delta c_4)_{A^4} \approx 0.9132288\,, && \sigma_1 \approx -0.4673613\,,\\
&(\delta c_2)_{A^4} \approx -4.3026050\,, && (\delta c_5)_{A^4} \approx -2.0134501\,, && \sigma_2 \approx 0.2171165\,,\\
&(\delta c_3)_{A^4} \approx -2.0134501\,, && (\delta c_6)_{A^4} \approx 1.4633393\,, && \sigma_3 \approx 1.6829758\,.
\end{aligned} \tag{5.7}$$

Elimination of α with (5.2) gives

$$\begin{aligned}
&c_1 \approx -2.1513026\,, && c_4 \approx 4.302605\,,\\
&c_2 \approx -4.302605\,, && c_5 \approx 0\,,\\
&c_3 \approx 0\,, && c_6 \approx 2.1513026\,.
\end{aligned} \tag{5.8}$$

These coefficients are not zero, so that the SFT effective action does not reproduce the abelian Born-Infeld action in a straightforward manner. Thus, we need to consider a field redefinition to put the effective action into the usual Born-Infeld form. To understand how this field redefinition works, it is useful to study the gauge transformation in the effective theory. Without directly computing this gauge transformation, we can write the general form that the transformation must take; the leading terms can be parameterized as

$$\delta A_\mu = \partial_\mu\lambda + g_{YM}^2(\varsigma_1 A^2\partial_\mu\lambda + \varsigma_2 A_\nu\partial_\mu A^\nu\lambda + \varsigma_3 A_\nu\partial^\nu A_\mu\lambda + \varsigma_4 A_\mu\partial\cdot A\lambda + \varsigma_5 A_\mu A_\nu\partial^\nu\lambda) + \mathcal{O}(\partial^3 A^2\lambda)\,. \tag{5.9}$$

The action (5.1) must be invariant under this gauge transformation. This gauge invariance imposes a number of *a priori* restrictions on the coefficients c_i, ς_i. When we vary the F^2 term in the effective action (4.1) the nonlinear part of (5.9) generates $\partial^3 A^3\lambda$ terms.

Gauge invariance requires that these terms cancel the terms arising from the linear gauge transformation of the $\partial^2 A^4$ terms in (5.1). This cancellation gives homogeneous linear equations for the parameters c_i and ς_i. The general solution of these equations depends on one free parameter γ:

$$\begin{aligned} c_1 &= -c_6 = -\gamma\,, & \varsigma_1 &= -\gamma\,, \\ c_2 &= -c_4 = -2\gamma\,, & \varsigma_5 &= -2\gamma\,, \\ c_3 &= c_5 = 0\,, & \varsigma_2 &= \varsigma_3 = \varsigma_4 = 0\,. \end{aligned} \tag{5.10}$$

The coefficients c_i calculated above satisfy these relations to 7 digits of precision. From the numerical values of the c_i's, we find

$$\gamma \approx 2.1513026 \pm 0.0000005\,. \tag{5.11}$$

We have thus found that the $\partial^2 A^4$ terms in the effective vector field action derived from SFT lie on a one-parameter family of possible combinations of terms which have a gauge invariance of the desired form. We can identify the degree of freedom associated with this parameter as arising from the existence of a family of field transformations with nontrivial terms at order A^3

$$\begin{aligned} \hat{A}_\mu &= A_\mu + g^2\gamma A^2 A_\mu\,, \\ \hat{\lambda} &= \lambda\,. \end{aligned} \tag{5.12}$$

We can use this field redefinition to relate a field $\hat{A}$ with the standard gauge transformation $\delta\hat{A}_\mu = \partial_\mu\hat{\lambda}$ to a field A transforming under (5.9) with ς_i and γ satisfying (5.10). Indeed, plugging this change of variables into

$$\begin{aligned} \delta\hat{A}_\mu &= \partial\hat{\lambda}\,, \\ S_{BI} &= -\frac{1}{4}\int dx \hat{F}^2 + \mathcal{O}(\hat{F}^3)\,. \end{aligned} \tag{5.13}$$

gives (5.9) and (5.1) with c_i, ς_i satisfying (5.10).

We have thus found that nonvanishing $\partial^2 A^4$ terms arise in the vector field effective action derived from SFT, but that these terms can be removed by a field redefinition. We would like to emphasize that the logic of this subsection relies upon using the fact that the effective vector field theory has a gauge invariance. The existence of this invariance constrains the action sufficiently that we can identify a field redefinition that puts the gauge transformation into standard form, without knowing in advance the explicit form of the gauge invariance in the effective theory. Knowing the field redefinition, however, in turn allows us to identify this gauge invariance explicitly. This interplay between field redefinitions and gauge invariance plays a key role in understanding higher-order terms in the effective action, which we explore further in the following subsection.

5.2 Gauge invariance and field redefinitions

In this subsection we discuss some aspects of the ideas of gauge invariance and field redefinitions in more detail. In the previous subsection, we determined a piece of the field redefinition relating the vector field A in the effective action derived from string field theory to the gauge field $\hat{A}$ in the Born-Infeld action by using the existence of a gauge invariance in the effective theory. The rationale for the existence of the field transformation from A to $\hat{A}$ can be understood based on the general theorem of the rigidity of the Yang-Mills gauge transformation [38, 39]. This theorem states that any deformation of the Yang-Mills gauge invariance can be mapped to the standard gauge invariance through a field redefinition. At the classical level this field redefinition can be expressed as

$$\begin{aligned} \hat{A}_\mu &= \hat{A}_\mu(A)\,, \\ \hat{\lambda} &= \hat{\lambda}(A,\lambda)\,. \end{aligned} \tag{5.14}$$

This theorem explains, for example, why noncommutative Yang-Mills theory, which has a complicated gauge invariance involving the noncommutative star product, can be mapped through the Seiberg-Witten map (field redefinition) to a gauge theory written in terms of a gauge field with standard transformation rules [12, 54]. Since in string field theory the parameter α' (which we have set to unity) parameterizes the deformation of the standard gauge transformation of A_μ, the theorem states that some field redefinition exists which takes the effective vector field theory arising from SFT to a theory which can be written in terms of the field strength $\hat{F}_{\mu\nu}$ and covariant derivative $\hat{D}_\mu$ of a gauge field $\hat{A}_\mu$ with the standard transformation rule.[5]

There are two ways in which we can make use of this theorem. Given the explicit expression for the effective action from SFT, one can assume that such a transformation exists, write the most general covariant action at the order of interest, and find a field redefinition which takes this to the effective action computed in SFT. Applying this approach, for example, to the $\partial^2 A^4$ terms discussed in the previous subsection, we would start with the covariant action $\hat{F}^2$, multiplied by an unknown overall coefficient ζ, write the field redefinition (5.12) in terms of the unknown γ, plug in the field redefinition, and match with the effective action (5.1), which would allow us to fix γ and $\zeta = -1/4$.

A more direct approach can be used when we have an explicit expression for the gauge invariance of the effective theory. In this case we can simply try to construct a field redefinition which relates this invariance to the usual Yang-Mills gauge invariance. When finding the field redefinition relating the deformed and undeformed theories, however, a further subtlety arises, which was previously encountered in related situations [14, 15]. Namely, there exists for any theory a class of trivial gauge invariances. Consider a theory with fields ϕ_i and action $S(\phi_i)$. This theory has trivial gauge transformations of the form

$$\delta\phi_i = \mu_{ij}\frac{\delta S}{\delta\phi_j} \tag{5.15}$$

where $\mu_{ij} = -\mu_{ji}$. Indeed, the variation of the action under this transformation is $\delta S = \mu_{ij}\frac{\delta S}{\delta\phi_i}\frac{\delta S}{\delta\phi_j} = 0$. These transformations are called trivial because they do not correspond to a constraint in the hamiltonian picture. The conserved charges associated with

[5] In odd dimensions there would also be a possibility of Chern-Simons terms

trivial transformations are identically zero. In comparing the gauge invariance of the effective action $S[A]$ to that of the Born-Infeld action, we need to keep in mind the possibility that the gauge invariances are not necessarily simply related by a field redefinition, but that the invariance of the effective theory may include additional terms of the form (5.15). In considering this possibility, we can make use of a theorem (theorem 3.1 of [55]), which states that under suitable regularity assumptions on the functions $\delta S/\delta\phi_i$ any gauge transformation that vanishes on shell can be written in the form (5.15). Thus, when identifying the field redefinition transforming the effective vector field A to the gauge field $\hat{A}$, we allow for the possible addition of trivial terms.

The benefit of the first method described above for determining the field redefinition is that we do not need to know the explicit form of the gauge transformation. Once the field redefinition is known we can find the gauge transformation law in the effective theory of A_μ up to trivial terms by plugging the field redefinition into the standard gauge transformation law of $\hat{A}_\mu$. In the explicit example of $\partial^2 A^4$ terms considered in the previous subsection we determined that the gauge transformation of the vector field A_μ is given by

$$\delta A_\mu = \partial_\mu\lambda - g_{\rm YM}^2\gamma(A^2\partial_\mu\lambda - 2A_\mu A_\nu\partial^\nu\lambda) \tag{5.16}$$

plus possible trivial terms which we did not consider. We have found the numerical value of γ in (5.11). If we had been able to directly compute this gauge transformation law, finding the field redefinition (5.12) would have been trivial. Unfortunately, as we shall see in a moment, the procedure for computing the higher-order terms in the gauge invariance of the effective theory is complicated to implement, which makes the second method less practical in general for determining the field redefinition. We can, however, at least compute the terms in the gauge invariance which are of order $A\lambda$ directly from the definition (2.5). Thus, for these terms the second method just outlined for computing the field redefinition can be used. We use this method in section 6.1 to compute the field redefinition including terms at order ∂A^2 and $\partial^2 A$ in the nonabelian theory.

Let us note that the field redefinition that makes the gauge transformation standard is not unique. There is a class of field redefinitions that preserves the gauge structure and mass-shell condition

$$\begin{aligned}\hat{A}'_\mu &= \hat{A}_\mu + T_\mu(\hat{F}) + \hat{D}_\mu\xi(\hat{A})\,,\\ \hat{\lambda}' &= \hat{\lambda} + \delta_{\hat{\lambda}}\xi(\hat{A}_\mu)\,.\end{aligned} \tag{5.17}$$

In this field redefinition $T_\mu(\hat{F})$ depends on $\hat{A}_\mu$ only through the covariant field strength and its covariant derivatives. The term ξ is a trivial (pure gauge) field redefinition, which is essentially a gauge transformation with parameter $\xi(A)$. The resulting ambiguity in the effective lagrangian has a field theory interpretation based on the equivalence theorem [56]. According to this theorem, different lagrangians give the same S-matrix elements if they are related by a change of variables in which both fields have the same gauge variation and satisfy the same mass-shell condition.

Let us now describe briefly how the different forms of gauge invariance arise in the world-sheet and string field theory approaches to computing the vector field action. We primarily carry out this discussion in the context of the abelian theory, although similar arguments can be made in the nonabelian case. In a world-sheet sigma model calculation one introduces the boundary interaction term

$$\oint_\gamma A_\mu \frac{dX^\mu}{d\tau} d\tau\,. \tag{5.18}$$

This term is explicitly invariant under

$$A_\mu \to A_\mu + \partial_\mu \lambda\,. \tag{5.19}$$

Provided that one can find a systematic method of calculation that respects this gauge invariance, the resulting effective action will possess this gauge invariance as well. This is the reason calculations such as those in [7, 8] give an effective action with the usual gauge invariance.

In the cubic SFT calculation, on the other hand, the gauge invariance is much more complicated. The original theory has an infinite number of gauge invariances, given by (2.5). We have fixed all but one of these gauge symmetries; the remaining symmetry comes from a gauge transformation that may change the field α, but which keeps all other auxiliary fields at zero. A direct construction of this gauge transformation in the effective theory of A_μ is rather complicated, but can be described perturbatively in three steps:

1. Make an SFT gauge transformation (in the full theory with an infinite number of fields) with the parameter
$$|\Lambda'\rangle = \frac{i}{\sqrt{2}}\lambda(x) b_{-1}|0\rangle\,. \tag{5.20}$$
This gauge transformation transforms α and A_μ as
$$\begin{aligned}\delta A_\mu &= \partial_\mu \lambda + i g_{YM}(\cdots)\,,\\ \delta\alpha &= \sqrt{2}\,\partial^2\lambda + i g_{YM}(\cdots)\,,\end{aligned} \tag{5.21}$$
and transforms all fields in the theory in a computable fashion.

2. The gauge transformation $|\Lambda'\rangle$ takes us away from the gauge slice we have fixed by generating fields associated with states containing c_0 at all higher levels. We now have to make a second gauge transformation with a parameter $|\Lambda''(\lambda)\rangle$ that will restore our gauge of choice. The order of magnitude of the auxiliary fields we have generated at higher levels is $\mathcal{O}(g\lambda\Phi)$. Therefore $|\Lambda''(\lambda)\rangle$ is of order $g\lambda\Phi$. Since we already used the gauge parameter at level zero, we will choose $|\Lambda''\rangle$ to have nonvanishing components only for massive modes. Then this gauge transformation does not change the massless fields linearly, so the contribution to the gauge transformation at the massless level will be of order $\mathcal{O}(g^2\lambda\Phi^2)$. The gauge transformation generated by $|\Lambda''(\lambda)\rangle$ can be computed as a perturbative expansion in g. Combining this with our original gauge transformation generated by $|\Lambda'\rangle$ gives us a new gauge transformation which transforms the massless fields linearly according to (5.21), but which also keeps us in our chosen gauge slice.

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3. In the third step we eliminate all the fields besides A_μ using the classical equations of motion. The SFT equations of motion are

$$Q_B|\Phi\rangle = -g\langle\Phi,\Phi|V_3\rangle\,. \tag{5.22}$$

The BRST operator preserves the level of fields; therefore, the solutions for massive fields and α in terms of A_μ will be of the form

$$\begin{aligned}\psi_{\mu_1,\ldots,\mu_n} &= \mathcal{O}(gA^2)\,,\\ \alpha &= \frac{1}{\sqrt{2}}\partial\cdot A + \mathcal{O}(gA^2)\end{aligned} \tag{5.23}$$

where $\psi_{\mu_1,\ldots,\mu_n}$ is a generic massive field. Using these EOM to eliminate the massive fields and α in the gauge transformation of A_μ will give terms of order $\mathcal{O}(g^2A^2)$.

To summarize, the gauge transformation in the effective theory for A_μ is of the form

$$\delta A_\mu = \partial_\mu\lambda + R_\mu(A,\lambda)\,, \tag{5.24}$$

where R_μ is a specific function of A and λ at order $g^2A^2\lambda$, which can in principle be computed using the method just described. In the nonabelian theory, there will also be terms at order $gA\lambda$ arising directly from the original gauge transformation $|\Lambda\rangle$; these terms are less complicated and can be computed directly from the cubic string field vertex.

In this subsection, we have discussed two approaches to computing the field redefinition which takes us from the effective action $S[A]$ to a covariant action written in terms of the gauge field $\hat{A}$, which should have the form of the Born-Infeld action plus derivative corrections. In the following sections we use these two approaches to check that various higher-order terms in the SFT effective action indeed agree with known terms in the Born-Infeld action, in both the abelian and nonabelian theories.

5.3 Terms of the form $\partial^4 A^4$

The goal of this subsection is to verify that after an appropriate field redefinition the ∂^4A^4 terms in the abelian effective action derived from SFT transform into the $\hat{F}^4 - \frac{1}{4}(\hat{F}^2)^2$ terms of the Born-Infeld action (including the correct constant factor of $(2\pi g_{YM})^2/8$). To demonstrate this, we use the first method discussed in the previous subsection. Since the total number of ∂^4A^4 terms is large we restrict attention to a subset of terms: namely those terms where indices on derivatives are all contracted together. These terms are independent from other terms at the same order in the effective action. By virtue of the equations of motion (5.2) the diagrams with α do not contribute to these terms. This significant simplification is the reason why we choose to concentrate on these terms. Although we only compute a subset of the possible terms in the effective action, however, we find that these terms are sufficient to fix both coefficients in the Born-Infeld action at order F^4.

The terms we are interested in have the general form

$$\begin{aligned}S^4_{(\partial\cdot\partial)A^4} = g^2\int d^{26}x\big(&d_1(\partial_\mu A_\lambda\partial^\mu A^\lambda)^2 + d_2\partial_\mu A_\lambda\partial_\nu A^\lambda\partial^\mu A_\sigma\partial^\nu A^\sigma +\\ &+ d_3A_\lambda\partial_\nu A^\lambda\partial_\mu A_\sigma\partial^\mu\partial^\nu A^\sigma + d_4\partial_\mu A_\lambda\partial_\nu A^\lambda A_\sigma\partial^\mu\partial^\nu A^\sigma +\\ &+ d_5A_\lambda A^\lambda\partial_\mu\partial_\nu A_\sigma\partial^\mu\partial^\nu A^\sigma + d_6A_\lambda\partial_\mu\partial_\nu A^\lambda A_\sigma\partial^\mu\partial^\nu A^\sigma\big)\,.\end{aligned} \tag{5.25}$$

The coefficients for these terms in the effective action are given by

$$d_i = \frac{1}{2}\mathcal{N}^2 \int_0^\infty d\tau e^\tau \operatorname{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right) P_i^{(4)}(A,B,C) \tag{5.26}$$

with

$$\begin{aligned}
P_1^{(4)} &= P_5^{(4)} = A_{11}^2 A_{00}^2 + B_{11}^2 B_{00}^2 + C_{11}^2 C_{00}^2 \,, \\
P_2^{(4)} &= P_6^{(4)} = A_{11}^2 \left(B_{00}^2 + C_{00}^2\right) + B_{11}^2 \left(A_{00}^2 + C_{00}^2\right) + C_{11}^2 \left(A_{00}^2 + B_{00}^2\right) , \\
P_3^{(4)} &= 4A_{11}^2 A_{00} \left(B_{00} + C_{00}\right) + 4B_{11}^2 B_{00} \left(A_{00} + C_{00}\right) + 4C_{11}^2 C_{00} \left(A_{00} + B_{00}\right) , \\
P_4^{(4)} &= 4A_{11}^2 B_{00} C_{00} + 4B_{11}^2 A_{00} C_{00} + 4C_{11}^2 A_{00} B_{00} \,.
\end{aligned} \tag{5.27}$$

Computation of the integrals gives us

$$\begin{aligned}
d_1 = d_5 &\approx 3.14707539, & d_3 &\approx 18.51562023\,, \\
d_2 = d_6 &\approx 2.96365920, & d_4 &\approx 0.99251621\,.
\end{aligned} \tag{5.28}$$

To match these coefficients with the BI action we need to write the general field redefinition to order $\partial^2 A^3$ (again, keeping only terms with all derivatives contracted)

$$\hat{A}_\mu = A_\mu + g^2\left(\gamma A^2 A_\mu + \alpha_1 A_\mu A_\sigma \partial^2 A^\sigma + \alpha_2 A^2 \partial^2 A_\mu + \alpha_3 A_\mu \partial_\lambda A_\sigma \partial^\lambda A^\sigma + \alpha_4 A_\sigma \partial_\lambda A_\mu \partial^\lambda A^\sigma\right). \tag{5.29}$$

Using the general theorem quoted in the previous subsection, we know that there is a field redefinition relating the action containing the terms (5.25) to a covariant action written in terms of a conventional field strength $\hat{F}$. The coefficients of $\hat{F}^2$ and $\hat{F}^3$ are already fixed, so the most generic action up to $\hat{F}^4$ is

$$\operatorname{Tr} \int dx \Big(-\frac{1}{4}\hat{F}^2 + g^2\left(a\hat{F}^4 + b\big(\hat{F}^2\big)^2\right) + O\big(\hat{F}^6\big)\Big). \tag{5.30}$$

We plug the change of variables (5.29) into this equation and collect $\partial^4 A^4$ terms with derivatives contracted together:

$$\begin{aligned}
g^2 \int d^{26}x \Big(&(\alpha_1 - \alpha_3 + 4b)(\partial_\mu A_\lambda \partial^\mu A^\lambda)^2 + (\alpha_1 + 2\alpha_2 - \alpha_4 + 2a)\partial_\mu A_\lambda \partial_\nu A^\lambda \partial^\mu A_\sigma \partial^\nu A^\sigma + \\
&+ (4\alpha_1 + 4\alpha_2 - 2\alpha_3 - \alpha_4) A_\lambda \partial_\nu A^\lambda \partial_\mu A_\sigma \partial^\mu \partial^\nu A^\sigma + \\
&+ (2\alpha_1 + 2\alpha_2 - \alpha_4)\partial_\mu A_\lambda \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma + \\
&+ \alpha_2 A_\lambda A^\lambda \partial_\mu \partial_\nu A_\sigma \partial^\mu \partial^\nu A^\sigma + \alpha_1 A_\lambda \partial_\mu \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma \Big).
\end{aligned} \tag{5.31}$$

The assumption that (5.25) can be written as (5.31) translates into a system of linear equations for a, b and $\alpha_1, \ldots \alpha_4$ with the right hand side given by $d_1, \ldots d_6$. This system is non-degenerate and has a unique solution

$$\begin{aligned}
\alpha_1 &= d_6 \approx 2.9636592\,, \\
\alpha_2 &= d_5 \approx 3.1470754\,, \\
\alpha_3 &= \frac{1}{2}(-d_3 + d_4 + 2d_5 + 2d_6) \approx -2.6508174\,, \\
\alpha_4 &= -d_4 + 2d_5 + 2d_6 \approx 11.2289530\,, \\
a &= \frac{1}{2}(d_2 - d_4 + d_6) \approx 2.4674011\,, \\
b &= \frac{1}{8}(2d_1 - d_3 + d_4 + 2d_5) \approx -0.6168503\,.
\end{aligned} \tag{5.32}$$

This determines the coefficients a and b in the effective action (5.30) to 8 digits of precision. These values agree precisely with those that we expect from the Born-Infeld action, which are given by

$$\begin{aligned}
a &= \frac{\pi^2}{4} \approx 2.4674011\,, \\
b &= -\frac{\pi^2}{16} \approx -0.6168502\,.
\end{aligned} \tag{5.33}$$

Thus, we see that after a field redefinition, the effective vector theory derived from string field theory agrees with Born-Infeld to order F^4, and correctly fixes the coefficients of both terms at that order. This calculation could in principle be continued to compute higher-derivative corrections to the Born-Infeld action of the form $\partial^6 A^4$ and higher, but we do not pursue such calculations further here.

Note that, assuming we know that the Born-Infeld action takes the form

$$S_{BI} = -T \int dx \sqrt{-\det\left(\eta^{\mu\nu} + T^{-\frac{1}{2}} F^{\mu\nu}\right)}\,. \tag{5.34}$$

with undetermined D-brane tension, we can fix $T = 1/(2\pi\alpha' g_{YM})^2$ from the coefficients at F^2 and F^4. We may thus think of the calculations done so far as providing another way to determine the D-brane tension from SFT.

5.4 Terms of the form A^{2n}

In the preceding discussion we have focused on terms in the effective action which are at most quartic in the vector field A_μ. It is clearly of interest to extend this discussion to terms of higher order in A. A complete analysis of higher-order terms, including all momentum dependence, involves considerable additional computation. We have initiated analysis of higher-order terms by considering the simplest class of such terms: those with no momentum dependence. As for the quartic terms of the form $(A^\mu A_\mu)^2$ discussed in section 4.2, we expect that all terms in the effective action of the form

$$(A^\mu A_\mu)^n \tag{5.35}$$

should vanish identically when all diagrams are considered. In this subsection we consider terms of the form (5.35). We find good numerical evidence that these terms indeed vanish, up to terms of the form A^{10}.

In section 4.2 we found strong numerical evidence that the term (5.35) vanishes for $n = 2$ by showing that the coefficient γ_+ in (4.13) approaches 0 in the level-truncation approximation. This A^4 term involves only one possible diagram. As n increases, the number of diagrams involved in computing A^{2n} increases exponentially, and the complexity of each diagram also increases, so that the primary method used in this paper becomes difficult to implement. To study the terms (5.35) we have used a somewhat different method, in which we directly truncate the theory by only including fields up to a fixed total oscillator level, and then computing the cubic terms for each of the fields below the desired level. This was the original method of level truncation used in [49] to compute the tachyon 4-point function, and in later work [50, 51] on level truncation on the problem of tachyon condensation. As discussed in section 3.3, the method we are using for explicitly calculating the quartic terms in the action involves truncating on the level of the intermediate state in the 4-point diagram, so that the two methods give the same answers. While level truncation on oscillators is very efficient for computing low-order diagrams at high level, however, level truncation on fields is more efficient for computing high-order diagrams at low level.

In [51], a recursive approach was used to calculate coefficients of ϕ^n in the effective tachyon potential from string field theory using level truncation on fields. Given a cubic potential

$$V = \sum_{i,j} d_{ij}\, \psi_i \psi_j + \sum_{i,j,k} g t_{ijk}\, \psi_i \psi_j \psi_k \tag{5.36}$$

for a finite number of fields $\psi_i, i = 1, \ldots, N$ at $p = 0$, the effective action for $a = \psi_1$ when all other fields are integrated out is given by

$$V_{\text{eff}}(a) = \sum_{n=2}^{\infty} \frac{1}{n} v^1_{n-1} a^n g^n \tag{5.37}$$

where v^i_n represents the summation over all graphs with n external a edges and a single external ψ^i, with no internal a's. The $v's$ satisfy the recursion relations

$$\begin{aligned} v^i_1 &= \delta^i_1 \\ v^i_n &= \frac{3}{2} \sum_{m=1}^{n-1} d^{ij}\, t_{jkl}\, \hat{v}^k_m \hat{v}^l_{n-m} \end{aligned} \tag{5.38}$$

where d^{ij} is the inverse matrix to d_{ij} and

$$\hat{v}^i_n = \begin{cases} 0, & i = 1 \text{ and } n > 1 \\ v^i_n, & \text{otherwise} \end{cases} \tag{5.39}$$

has been defined to project out internal a edges.

We have used these relations to compute the effective action for A_μ at $p = 0$. We computed all quadratic and cubic interactions between fields up to level 8 associated with states which are scalars in 25 of the space-time dimensions and which include an arbitrary number of matter oscillators a^{25}_{-n}. Plugging the resulting quadratic and cubic coefficients into the recursion relations (5.38) allows us to compute the coefficients $c_{2n} = v^1_{2n-1}/2n$ in the effective action for the gauge field A_μ

$$\sum_{n=1}^{\infty} -c_{2n} g^n (A^\mu A_\mu)^n \tag{5.40}$$

for small values of n . We have computed these coefficients up to $n = 7$ at different levels of field truncation up to $L = 8$. The results of this computation are given in table 2 up to $n = 5$, including the predicted value at $L = \infty$ from a $1/L$ fit to the data at levels 2, 4, 6 and 8. The results in table 2 indicate that, as expected, all coefficients c_{2n} will vanish when the level is taken to infinity. The initial contribution at level 2 is canceled to within 0.6% for terms A^4, within 0.8% for terms A^6, within 4% for terms A^8, and within 7% for terms A^{10}. It is an impressive success of the level-truncation method that for c_{10}, the cancellation predicted by the $1/L$ expansion is so good, given that the coefficients computed in level truncation *increase* until level $L = 8$. We have also computed the coefficients for larger values of n, but for $n > 5$ the numerics are less compelling. Indeed, the approximations to the coefficients c_{12} and beyond continue to grow up to level 8. We expect that a good prediction of the cancellation of these higher-order terms would require going to higher level.

Level	c_4	c_6	c_8	c_{10}
2	0.200	1.883	6.954	28.65
4	0.097	1.029	6.542	37.49
6	0.063	0.689	5.287	37.62
8	0.046	0.517	4.325	34.18
∞	0.001	0.014	-0.229	1.959

Table 2: Coefficients of A^{2n} at various levels of truncation.

The results found here indicate that the method of level truncation in string field theory seems robust enough to correctly compute higher-order terms in the vector field effective action. Computing terms with derivatives at order A^6 and beyond would require some additional work, but it seems that a reasonably efficient computer program should be able to do quite well at computing these terms, even to fairly high powers of A.

6. The nonabelian Born-Infeld action

We now consider the theory with a nonabelian gauge group. As we discussed in section 2.3, the first term beyond the Yang-Mills action in the nonabelian analogue of the Born-Infeld action has the form $\mathrm{Tr}\,\hat{F}^3$. As in the previous section, we expect that a field redefinition is necessary to get this term from the effective nonabelian vector field theory derived from SFT. In this section we compute the terms in the effective vector field theory to orders $\partial^3 A^3$ and $\partial^2 A^4$, and we verify that after a field redefinition these terms reproduce the corresponding pieces of the $\hat{F}^3$ term, with the correct coefficients. In section 6.1 we analyze $\partial^3 A^3$ terms, and in subsection 6.2 we consider the $\partial^2 A^4$ terms.

6.1 $\partial^3 A^3$ terms

In section 4.2 we showed that the terms of the form ∂A^3 in the nonabelian SFT effective action for A contribute to the $\hat{F}^2$ term after a field redefinition. We now consider terms at order $\partial^3 A^3$. Recall from (4.6) and (4.8) that the full effective action for α and A at this order is given by

$$\begin{aligned}\check{S}_{(A+\alpha)^3}[A,\alpha] = ig_{YM} \int dx\, \mathrm{Tr}\Big(\frac{1}{6}\big(\partial_\lambda \tilde{A}^\mu \partial_\mu \tilde{A}^\nu \partial_\nu \tilde{A}^\lambda - \partial_\nu \tilde{A}^\mu \partial_\lambda \tilde{A}^\nu \partial_\mu \tilde{A}^\lambda\big) - \partial_\mu \tilde{A}_\nu [\tilde{A}^\mu, \tilde{A}^\nu] + \\ + \frac{1}{2}[\tilde{A}_\nu, \partial^\lambda \tilde{A}_\mu]\partial^\mu \partial^\nu \tilde{A}_\lambda + \tilde{A}^\mu\, [\partial_\mu \tilde{\alpha}; \tilde{\alpha}]\Big) \qquad (6.1)\end{aligned}$$

where $\tilde{A}_\mu = \exp(-\frac{1}{2}V_{00}^{11}\partial^2)A_\mu$, and similarly for $\tilde{\alpha}$. After eliminating α from (6.1) using the equation of motion obtained from (4.1) and integrating by parts to remove terms containing ∂A, we find that the complete set of terms at order $\partial^3 A^3$ is given by

$$\begin{aligned}S_{A^3}^{[3]}[A] = ig_{YM}\int dx\, \mathrm{Tr}\Big(\frac{2}{3}(\partial_\lambda A^\mu \partial_\mu A^\nu \partial_\nu A^\lambda - \partial_\nu A^\mu \partial_\lambda A^\nu \partial_\mu + A^\lambda) \qquad (6.2) \\ + \frac{1}{2}V_{00}^{11}\left(\partial_\mu \partial^2 A_\nu [A^\mu, A^\nu] + \partial_\mu A_\nu [\partial^2 A^\mu, A^\nu] + \partial_\mu A_\nu [A^\mu, \partial^2 A^\nu]\right)\Big)\,.\end{aligned}$$

Note that unlike the quartic terms in A, our expressions for these terms are exact.

Let us now consider the possible terms that we can get after the field redefinition to the field $\hat{A}$ with standard gauge transformation rules. Following the analysis of [47], we write the most general covariant action to order $\hat{F}^3$ (keeping D at order $F^{1/2}$ as discussed above)

$$-\frac{1}{4}\hat{F}^2 + ig_{YM} a \hat{F}^3 + \chi \hat{D}_\sigma \hat{F}^{\sigma\mu} \hat{D}^\nu \hat{F}_{\nu\mu} + \mathcal{O}(\hat{F}^4)\,, \qquad (6.3)$$

where

$$\hat{D}_\mu = \partial_\mu - ig_{YM}[\hat{A}_\mu, \cdot\,]\,. \qquad (6.4)$$

The action (6.3) is not invariant under field redefinitions which keep the gauge invariance unchanged. Under the field redefinition

$$\hat{A}'_\mu = \hat{A}_\mu + \upsilon \hat{D}_\sigma \hat{F}^\sigma_{\ \mu}\,. \qquad (6.5)$$

we have

$$\begin{aligned} a' &= a\,, \\ \chi' &= \chi - \upsilon\,. \qquad (6.6)\end{aligned}$$

Thus, the coefficient a is defined unambiguously, while χ can be set to any chosen value by a field redefinition.

Just as we have an exact formula for the cubic terms in the SFT action, we can also compute the gauge transformation rule exactly to quadratic order in A using (2.5). After

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some calculation, we find that the gauge variation for A_μ to order $A^2\lambda$ is given by (before integrating out α)

$$\delta A_\mu = \partial_\mu\lambda - ig_{YM}\Big([A_\mu,\lambda]_\star - [\partial_\mu A_\nu,\partial^\nu\lambda]_\star + [A^\nu,\partial_\mu\partial_\nu\lambda]_\star + \frac{1}{\sqrt{2}}[\partial_\mu B,\lambda]_\star - \frac{1}{\sqrt{2}}[B,\partial_\mu\lambda]_\star\Big)\,. \tag{6.7}$$

where $B = \alpha - \frac{1}{\sqrt{2}}\partial_\mu A^\mu$ as in section 4.1. The commutators are taken with respect to the product

$$f(x)\star g(x) = f(x)e^{-V^{11}_{00}(\overleftarrow{\partial}^2+\overleftarrow{\partial}\cdot\overrightarrow{\partial}+\overrightarrow{\partial}^2)}g(x)\,. \tag{6.8}$$

The equation of motion for α at leading order is simply $B=0$. Eliminating α we therefore have

$$\delta A_\mu = \partial_\mu\lambda - ig_{YM}\Big([A_\mu,\lambda]_\star + [\partial^\nu\lambda,\partial_\mu A_\nu]_\star + [A^\nu,\partial_\mu\partial_\nu\lambda]_\star\Big)\,. \tag{6.9}$$

We are interested in considering the terms at order $\partial^2 A\lambda$ in this gauge variation. Recall that in section 5.2 we observed that the gauge transformation may include extra trivial terms which vanish on shell. Since the leading term in the equation of motion for A arises at order $\partial^2 A$, it is possible that (6.9) may contain a term of the form

$$\delta A_\mu = \rho\,[\lambda,\partial^2 A_\mu - \partial_\mu\partial\cdot A] + \mathcal{O}(\lambda A^2) \tag{6.10}$$

in addition to a part which can be transformed into the standard nonabelian gauge variation through a field redefinition. Thus, we wish to consider the one-parameter family of gauge transformations

$$\begin{aligned}\delta A = \partial_\mu\lambda - ig_{YM}\big([A_\mu,\lambda] - V^{11}_{00}[\partial^2 A_\mu,\lambda] - V^{11}_{00}[\partial_\nu A_\mu,\partial^\nu\lambda] - V^{11}_{00}[A_\mu,\partial^2\lambda] +\\ +\rho\,[\lambda,\partial^2 A_\mu - \partial_\mu\partial\cdot A] + \mathcal{O}(\lambda A^2,\lambda\partial^4 A)\big)\,,\end{aligned} \tag{6.11}$$

where ρ is an as-yet undetermined constant. We now need to show, following the second method discussed in subsection 5.2, that there exists a field redefinition which takes a field A with action (6.2) and a gauge transformation of the form (6.11) to a gauge field $\hat{A}$ with an action of the form (6.3) and the standard nonabelian gauge transformation rule.

The leading terms in the field redefinition can be parameterized as

$$\begin{aligned}\hat{A}_\mu &= A_\mu + v_1\partial_\mu\partial\cdot A + v_2\partial^2 A_\mu + ig_{YM}\big(v_3[A_\sigma,\partial_\mu A^\sigma] + v_4[A_\mu,\partial\cdot A] + v_5[\partial_\sigma A_\mu,A^\sigma]\big)\,,\\ \hat{\lambda} &= \lambda + v_6\partial^2\lambda + ig_{YM}\big(v_7[\partial\cdot A,\lambda] + v_8[A_\sigma,\partial^\sigma\lambda]\big)\,.\end{aligned} \tag{6.12}$$

The coefficient v_1 can be chosen arbitrarily through a gauge transformation, so we simply choose $v_1 = -v_2$. The requirement that the r.h.s. of (6.12) varied with (6.11) and rewritten in terms of $\hat{A}$, $\hat{\lambda}$ gives the standard transformation law for $\hat{A}$, $\hat{\lambda}$ up to terms of order $\mathcal{O}(\hat{\lambda}\hat{A}^2)$ gives a system of linear equations with solutions depending on one free parameter v.

$$\begin{aligned} v_2 &= -v_1 = v\,, & \rho &= V^{11}_{00}\,,\\ v_3 &= 1-\frac{1}{2}V^{11}_{00}+v\,, & v_6 &= 0\,,\\ v_4 &= -V^{11}_{00}+v\,, & v_7 &= V^{11}_{00}\,,\\ v_5 &= -V^{11}_{00}+2v\,, & v_8 &= \frac{1}{2}V^{11}_{00}\,.\end{aligned} \tag{6.13}$$

It is easy to see that the parameter υ generates the field redefinition (6.5). For simplicity, we set $\upsilon = 0$. The field redefinition is then given by

$$\hat{A}_\mu = A_\mu - ig_{YM}\left(\left(\frac{1}{2}V_{00}^{11} - 1\right)[A_\sigma, \partial_\mu A^\sigma] + V_{00}^{11}[A_\mu, \partial \cdot A] + V_{00}^{11}[\partial_\sigma A_\mu, A^\sigma]\right). \quad (6.14)$$

We can now plug in the field redefinition (6.14) into the action (6.3) and compare with the $\partial^3 A^3$ term in the SFT effective action (6.2). We find agreement when the coefficients in (6.3) are given by

$$a = \frac{2}{3}, \qquad \chi = 0. \quad (6.15)$$

Thus, we have shown that the terms of order $\partial^3 A^3$ in the effective nonabelian vector field action derived from SFT are in complete agreement with the first nontrivial term in the nonabelian analogue of the Born-Infeld theory, including the overall constant. Note that while the coefficient of a agrees with that in (2.28), the condition $\chi = 0$ followed directly from our choice $\upsilon = 0$; other choices of υ would lead to other values of χ, which would be equivalent under the field redefinition (6.5).

6.2 $\partial^2 A^4$ terms

In the abelian theory, the $\partial^2 A^4$ terms disappear after the field redefinition. In the nonabelian case, however, the term proportional to $\hat{F}^3$ contains terms of the form $\partial^2 \hat{A}^4$. In this subsection, we show that these terms are correctly reproduced by string field theory after the appropriate field redefinition. Just as in section 5.3, for simplicity we shall concentrate on the $\partial^2 A^4$ terms where the Lorentz indices on derivatives are contracted together.

The terms of interest in the effective nonabelian vector field action can be written in the form

$$S_{A^4}^{[2]} = g_{YM}^2 \int d^{26}x \Big(f_1 \partial_\sigma A_\mu A^\mu \partial^\sigma A_\nu A^\nu + f_2 \partial_\sigma A_\mu A^\mu A^\nu \partial^\sigma A_\nu + f_3 A^\mu \partial_\sigma A_\mu A_\nu \partial^\sigma A^\nu + \\ + f_4 \partial_\sigma A_\mu \partial^\sigma A^\mu A_\nu A^\nu + f_5 \partial_\sigma A_\mu \partial^\sigma A_\nu A^\mu A^\nu + f_6 \partial_\sigma A_\mu A^\nu \partial^\sigma A_\mu A^\nu \Big) \quad (6.16)$$

where the coefficients f_i will be determined below. The coefficients of the terms in the field redefinition which are linear and quadratic in A were fixed in the previous subsection. The relevant terms in the field redefinition for computing the terms we are interested in here are generic terms of order A^3 with no derivatives, as well as those from (6.14) that do not have ∂_μ's contracted with A_μ's. Keeping only these terms we can parametrize the field redefinition as

$$\hat{A}_\mu = A_\mu + ig_{YM}\left(1 - \frac{V_{00}^{11}}{2}\right)[A_\sigma, \partial_\mu A^\sigma] + g_{YM}^2\left(\rho_1 A_\sigma A_\mu A^\sigma + \rho_2 A^2 A_\mu + \rho_3 A_\mu A^2\right). \quad (6.17)$$

In the abelian case this formula reduces to (5.12) with $\rho_1 + \rho_2 + \rho_3 = 2\gamma$. Plugging this field redefinition into the action

$$\hat{S}[\hat{A}_\mu] = \int \mathrm{Tr}\left(-\frac{1}{4}\hat{F}^2 + \frac{2i}{3}g_{YM}\hat{F}^3 + \mathcal{O}(\hat{F}^4)\right). \quad (6.18)$$

and collecting $\partial^2 A^4$ terms with indices on derivatives contracted together we get

$$g_{YM}^2 \int dx \Bigg[\left(\frac{1}{2}V_{00}^{11} - 1 - \rho_3\right) \partial_\sigma A_\mu A^\mu \partial^\sigma A_\nu A^\nu - (\rho_2 + \rho_3 + V_{00}^{11}) \partial_\sigma A_\mu A^\mu A_\nu \partial^\sigma A^\nu + \left(\frac{1}{2}V_{00}^{11} - 1 - \rho_2\right) A_\mu \partial^\sigma A^\mu A_\nu \partial^\sigma A^\nu - (\rho_2 + \rho_3) \partial_\sigma A_\mu \partial^\sigma A^\mu A_\nu A^\nu + (2 - 2\rho_1) \partial_\sigma A_\mu \partial^\sigma A_\nu A^\mu A^\nu - \rho_1 \partial_\sigma A_\mu A_\nu \partial^\sigma A^\mu A_\nu \Bigg] . \tag{6.19}$$

Comparing (6.19) and (6.16) we can write the unknown coefficients in the field redefinition in terms of the f_i's through

$$\rho_1 = -f_6 \,, \qquad \rho_2 = \rho_3 = -\frac{1}{2} f_4 \,. \tag{6.20}$$

We also find a set of constraints on the f_i's which we expect the values computed from the SFT calculation to satisfy, namely

$$f_1 - \frac{1}{2} f_4 = -1 + \frac{1}{2} V_{00}^{11} \,, \qquad f_2 - f_4 = -V_{00}^{11} \,, \qquad f_5 - 2f_6 = 2 \,. \tag{6.21}$$

On the string field theory side the coefficients f_i are given by

$$f_i = \frac{1}{2} \mathcal{N}^2 \int_0^\infty d\tau e^\tau \, \mathrm{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) P_{\partial^2 A^4, i}(A, B, C) \tag{6.22}$$

where, in complete analogy with the previous examples, the polynomials $P_{\partial^2 A^4, i}$ derived from (3.25) and (3.26) have the form

$$\begin{aligned}
P_{\partial^2 A^4,1} &= -2\big(A_{11}^2 B_{00} + C_{11}^2 B_{00}\big) \,, & P_{\partial^2 A^4,4} &= -4\big(A_{11}^2 A_{00} + C_{11}^2 C_{00}\big) \,, \\
P_{\partial^2 A^4,2} &= -4\big(A_{11}^2 C_{00} + C_{11}^2 A_{00}\big) \,, & P_{\partial^2 A^4,5} &= -4B_{11}^2 \big(A_{00} + C_{00}\big) \,, \\
P_{\partial^2 A^4,3} &= -2\big(A_{11}^2 B_{00} + C_{11}^2 B_{00}\big) \,, & P_{\partial^2 A^4,6} &= -4B_{11}^2 B_{00} \,.
\end{aligned} \tag{6.23}$$

Numerical computation of the integrals gives

$$\begin{aligned}
f_1 &\approx -2.2827697 \,, & f_4 &\approx -2.0422916 \,, \\
f_2 &\approx -1.5190433 \,, & f_5 &\approx -2.5206270 \,, \\
f_3 &\approx -2.2827697 \,, & f_6 &\approx -2.2603135 \,.
\end{aligned} \tag{6.24}$$

As one can easily check, the relations (6.21) are satisfied with high accuracy. This verifies that the $\partial^2 A^4$ terms we have computed in the effective vector field action are in agreement with the $\hat{F}^3$ term in the nonabelian analogue of the Born-Infeld action.

7. Conclusions

In this paper we have computed the effective action for the massless open string vector field by integrating out all other fields in Witten's cubic open bosonic string field theory. We have calculated the leading terms in the off-shell action $S[A]$ for the massless vector field

A_μ, which we have transformed using a field redefinition into an action $\hat{S}[\hat{A}]$ for a gauge field $\hat{A}$ which transforms under the standard gauge transformation rules. For the abelian theory, we have shown that the resulting action agrees with the Born-Infeld action to order $\hat{F}^4$, and that zero-momentum terms vanish to order A^{10}. For the nonabelian theory, we have shown agreement with the nonabelian effective vector field action previously computed by world-sheet methods to order $\hat{F}^3$. These results demonstrate that string field theory provides a systematic approach to computing the effective action for massless string fields. In principle, the calculation in this paper could be continued to determine higher-derivative corrections to the abelian Born-Infeld action and higher-order terms in the nonabelian theory.

As we have seen in this paper, comparing the string field theory effective action to the effective gauge theory action computed using world-sheet methods is complicated by the fact that the fields defined in SFT are related through a nontrivial field redefinition to the fields defined through world-sheet methods. In particular, the massless vector field in SFT has a nonstandard gauge invariance, which is only related to the usual Yang-Mills gauge invariance through a complicated field redefinition. This is a similar situation to that encountered in noncommutative gauge theories, where the gauge field in the noncommutative theory—whose gauge transformation rule is nonstandard and involves the noncommutative star product—is related to a gauge field with conventional transformation rules through the Seiberg-Witten map. In the case of noncommutative Yang-Mills theories, the structure of the field redefinition is closely related to the structure of the gauge-invariant observables of the theory, which in that case are given by open Wilson lines [57]. A related construction recently appeared in [58], where a field redefinition was used to construct matrix objects transforming naturally under the D4-brane gauge field in a matrix theory of D0-branes and D4-branes. An important outstanding problem in string field theory is to attain a better understanding of the observables of the theory (some progress in this direction was made in [59, 60]). It seems likely that the problem of finding the field redefinition between SFT and world-sheet fields is related to the problem of understanding the proper observables for open string field theory.

While we have focused in this paper on calculations in the bosonic theory, it would be even more interesting to carry out analogous calculations in the supersymmetric theory. There are currently several candidates for an open superstring field theory, including the Berkovits approach [61] and the (modified) cubic Witten approach [62, 63, 64]. (See [65] for further references and a comparison of these approaches.) In the abelian case, a superstring calculation should again reproduce the Born-Infeld action, including all higher-derivative terms. In the nonabelian case, it should be possible to compute all the terms in the nonabelian effective action. Much recent work has focused on this nonabelian action, and at this point the action is constrained up to order F^6 [11]. It would be very interesting if some systematic insight into the form of this action could be gained from SFT.

The analysis in this paper also has an interesting analogue in the closed string context. Just as the Yang-Mills theory describing a massless gauge field can be extended to a full stringy effective action involving the Born-Infeld action plus derivative corrections, in the closed string context the Einstein theory of gravity becomes extended to a stringy effective action containing higher order terms in the curvature. Some terms in this action have been computed, but they are not yet understood in the same systematic sense as the abelian Born-Infeld theory. A tree-level computation in closed string field theory would give an effective action for the multiplet of massless closed string fields, which should in principle be mapped by a field redefinition to the Einstein action plus higher-curvature terms [14]. Lessons learned about the nonlocal structure of the effective vector field theory discussed in this paper may have interesting generalizations to these nonlocal extensions of standard gravity theories.

Another direction in which it would be interesting to extend this work is to carry out an explicit computation of the effective action for the tachyon in an unstable brane background, or for the combined tachyon-vector field effective action. Some progress on the latter problem was made in [15]. Because the mass-shell condition for the tachyon is $p^2 = 1$, it does not seem to make any sense to consider an effective action for the tachyon field, analogous to the Born-Infeld action, where terms of higher order in p are dropped. Indeed, it can be shown that when higher-derivative terms are dropped, any two actions for the tachyon which keep only terms $\partial^k \phi^{m+k}, m \geq 0$, can be made perturbatively equivalent under a field redefinition (which may, however, have a finite radius of convergence in p). Nonetheless, a proposal for an effective tachyon + vector field action of the form

$$S = V(\phi)\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi)} \tag{7.1}$$

was given in [17, 18, 19] (see also [16]). Quite a bit of recent work has focused on this form of effective action, and there seem to be many special properties for this action with particular forms of the potential function $V(\phi)$. In particular, it was shown in [66], following earlier work [68], that with the correct choice of potential the action (7.1) correctly describes physics around a classical exponentially dependent tachyon background satisfying the classical equations of motion. This gives a well-defined regime in which the action (7.1) makes sense as a Born-Infeld type action. It would be very interesting to explicitly construct the tachyon-vector action using the methods of this paper. A particularly compelling question related to this action is that of closed string radiation during the tachyon decay process. In order to understand this radiation process, it is necessary to understand back-reaction on the decaying D-brane [67], which in the open string limit corresponds to the computation of loop diagrams. Recent work [20] indicates that for the superstring, SFT loop diagrams on an unstable Dp-brane with $p < 7$ should be finite, so that it should be possible to include loop corrections in the effective tachyon action in such a theory. The resulting effective theory should shed light on the question of closed string radiation from a decaying D-brane.

Ultimately, however, it seems that the most important questions which may be addressed using the type of effective field theory computed in this paper have to do with the nonlocal nature of string theory. The full effective action for the massless fields on a D-brane, given by the Born-Infeld action plus derivative corrections, or by the nonabelian vector theory on multiple D-branes, has a highly nonlocal structure. Such nonlocal actions are very difficult to make sense of from the point of view of conventional quantum field theory. Nonetheless, there is important structure hidden in the nonlocality of open string theory. For example, the instability associated with contact interactions between two parts of a D-brane world-volume which are separated on the D-brane but coincident in space-time is very difficult to understand from the point of view of the nonlocal theory on the D-brane, but is implicitly contained in the classical nonlocal D-brane action. At a more abstract level, we expect that in any truly background-independent description of quantum gravity, space-time geometry and topology will be an emergent phenomenon, not manifest in any fundamental formulation of the theory. A nongeometric formulation of the theory is probably necessary for addressing questions of cosmology and for understanding very early universe physics before the Planck time. It seems very important to develop new tools for grappling with such issues, and it may be that string field theory may play an important role in developments in this direction. In particular, the way in which conventional gauge theory and the nonlocal structure of the D-brane action is encoded in the less geometric variables of open string field theory may serve as a useful analogue for theories in which space-time geometry and topology emerge from a nongeometric underlying theory.

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A. Neumann coefficients

In this appendix we give explicit expressions for and properties of the Neumann coefficients that we use throughout this paper. First we define coefficients A_n and B_n by the series expansions

$$\left(\frac{1+iz}{1-iz}\right)^{1/3} = \sum_{n\,\text{even}} A_n z^n + i \sum_{n\,\text{odd}} A_n z^n \,, \tag{A.1}$$

$$\left(\frac{1+iz}{1-iz}\right)^{2/3} = \sum_{n\,\text{even}} B_n z^n + i \sum_{n\,\text{odd}} B_n z^n \,. \tag{A.2}$$

In terms of A_n and B_n we define the coefficients $N_{mn}^{r,\pm s}$ as follows:

$$\begin{aligned}
N_{nm}^{r,\pm r} &= \frac{1}{3(n \pm m)} \begin{cases} (-1)^n (A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z}\,, \quad m \neq n \\ 0 & m+n \in 2\mathbb{Z}+1 \end{cases}, \\
N_{nm}^{r,\pm(r+1)} &= \frac{1}{6(n \pm m)} \begin{cases} (-1)^{n+1} (A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z}\,, \quad m \neq n \\ \sqrt{3}\,(A_n B_m \mp B_n A_m) & m+n \in 2\mathbb{Z}+1 \end{cases}, \\
N_{nm}^{r,\pm(r-1)} &= \frac{1}{6(n \mp m)} \begin{cases} (-1)^{n+1} (A_n B_m \mp B_n A_m) & m+n \in 2\mathbb{Z}\,, \quad m \neq n \\ -\sqrt{3}\,(A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z}+1 \end{cases}.
\end{aligned} \tag{A.3}$$

The coefficients V^{rs}_{mn} are then given by

$$
\begin{aligned}
V^{rs}_{nm} &= \sqrt{mn}\,\left(N^{r,s}_{nm} + N^{r,-s}_{nm}\right) && m \neq n,\, m,n > 0\,, && \text{(A.4a)}\\
V^{rr}_{nn} &= \frac{1}{3}\left(2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - (-1)^n - A_n^2\right), && n \neq 0\,, && \text{(A.4b)}\\
V^{r(r+1)}_{nn} &= V^{r(r+2)}_{nn} = -\frac{1}{2}\left((-1)^n + V^{rr}_{nn}\right) && n \neq 0\,, && \text{(A.4c)}\\
V^{rs}_{0n} &= \sqrt{2n}\,\left(N^{r,s}_{0n} + N^{r,-s}_{0n}\right) && n \neq 0\,, && \text{(A.4d)}\\
V^{rr}_{00} &= -\ln\left(\frac{27}{16}\right). && && \text{(A.4e)}
\end{aligned}
$$

The analogous expressions for the ghost Neumann coefficients are

$$
\begin{aligned}
\mathcal{N}^{r,\pm r}_{nm} &= \frac{1}{3(n\pm m)}\begin{cases}(-1)^{n+1}(B_nA_m \pm A_nB_m) & m+n \in 2\mathbb{Z}\,,\quad m\neq n\\ 0 & m+n\in 2\mathbb{Z}+1\end{cases},\\
\mathcal{N}^{r,\pm(r+1)}_{nm} &= \frac{1}{6(n\pm m)}\begin{cases}(-1)^n\,(B_nA_m \pm A_nB_m) & m+n \in 2\mathbb{Z}\,,\quad m\neq n\\ -\sqrt{3}\,(B_nA_m \mp A_nB_m) & m+n\in 2\mathbb{Z}+1\end{cases},\\
\mathcal{N}^{r,\pm(r-1)}_{nm} &= \frac{1}{6(n\mp m)}\begin{cases}(-1)^n(B_nA_m \mp A_nB_m) & m+n \in 2\mathbb{Z}\,,\quad m\neq n\\ \sqrt{3}\,(B_nA_m \pm A_nB_m) & m+n\in 2\mathbb{Z}+1\end{cases}. \qquad \text{(A.5)}
\end{aligned}
$$

Observe that the ghost formulae (A.5) are related to matter ones (A.3) by $A_m \to -B_m$, $B_m \to A_m$. The ghost Neumann coefficients are expressed via $\mathcal{N}^{rs}_{nm}$ as

$$
\begin{aligned}
X^{rs}_{nm} &= m\,(\mathcal{N}^{r,s}_{nm} + \mathcal{N}^{r,-s}_{nm}) && m\neq n,\, m>0\,, && \text{(A.6a)}\\
X^{rr}_{nn} &= -\frac{2}{3}(-1)^nA_nB_n + \frac{1}{3}\left(2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - (-1)^n - A_n^2\right) && n\neq 0\,, && \text{(A.6b)}\\
X^{rs}_{nn} &= X^{rs}_{nn} = -\frac{1}{2}\left((-1)^n + X^{rr}_{nn}\right), && r\neq s, n\neq 0\,, && \text{(A.6c)}
\end{aligned}
$$

The exponential in the vertex $\langle V_3|$ does not contain X_{n0}, so we have not included an expression for this coefficient; alternatively, we can simply define this coefficient to vanish and include c_0 in the exponential in $\langle V_3|$.

Now we describe some algebraic properties satisfied by V^{rs} and X^{rs}. Define $M^{rs}_{mn} = CV^{rs}$, $\mathcal{M}^{rs}_{mn} = \sqrt{\frac{n}{m}}CX^{rs}_{mn}$. The matrices M and $\mathcal{M}$ satisfy symmetry and cyclicity properties

$$
\begin{aligned}
M^{r+1\,s+1} &= M^{rs}\,, & \mathcal{M}^{r+1\,s+1} &= \mathcal{M}^{rs}\,, && \text{(A.7a)}\\
(M^{rs})^T &= M^{rs}\,, & (\mathcal{M}^{rs})^T &= \mathcal{M}^{rs}\,, && \text{(A.7b)}\\
CM^{rs}C &= M^{sr}\,, & C\mathcal{M}^{rs}C &= \mathcal{M}^{sr}\,. && \text{(A.7c)}
\end{aligned}
$$

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This reduces the set of independent matter Neumann matrices to M^{11}, M^{12}, M^{21} and similarly for ghosts. These matrices commute and in addition satisfy

$$M^{11} + M^{12} + M^{21} = -1\,, \qquad \mathcal{M}^{11} + \mathcal{M}^{12} + \mathcal{M}^{21} = -1\,, \tag{A.8a}$$

$$M^{12}M^{21} = M^{11}(M^{11} + 1)\,, \qquad \mathcal{M}^{12}\mathcal{M}^{21} = \mathcal{M}^{11}(\mathcal{M}^{11} \quad -1)\,. \tag{A.8b}$$

These relations imply that there is only one independent Neumann matrix.

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